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DISSIPATIVE, FORCED TURBULENCE
IN TWO-DIMENSIONAL MAGNETOHYDRODYNAMICS

by

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ABSTRACT

The equations of motion for turbulent two-dimensional magnetohydrodynamic flows are solved in the presence of finite viscosity and resistivity, for the case in which external forces (mechanical and/or magnetic) act on the fluid. The goal is to verify the existence of a magnetohydrodynamic dynamo effect which is represented mathematically by a substantial back-transfer of mean square vector potential to the longest allowed Fourier wavelengths. External forces consisting of a random part plus a fraction of the value at the previous time step are employed, after the manner of Lilly for the Navier-Stokes case. The regime explored is that for which the mechanical and magnetic Reynolds numbers are in the region of 100 to 1000. The conclusions are that mechanical forcing terms alone cannot lead to dynamo action, but that dynamo action can result from either magnetic forcing terms or from both mechanical and magnetic forcing terms simultaneously. Most real physical cases seem most accurately modelled by the third situation. The spatial resolution of the 32×32 calculation is not adequate to test accurately the predictions of the spectral power laws previously arrived at on the basis of the assumption of simultaneous cascades of energy and vector potential. Some speculations are offered concerning possible relations between turbulent cascades and the "disruptive instability."

1. INTRODUCTION

This is intended as the final paper in a series of three dealing with incompressible two-dimensional magnetohydrodynamic turbulence. (A more exhaustive bibliography has been given in the first two of these: Fyfe and Montgomery 1975; Fyfe, Joyce, and Montgomery 1976.) In the first two papers, non-dissipative equilibrium predictions were algebraically derived and numerically tested, and an inverse vector potential cascade was predicted for the case of finite dissipation and external forcing. Our purpose here is to present the results of a numerical investigation of driven MHD turbulence for two-dimensional incompressible flow in the presence of finite conductivity and viscosity.

Considerable interest in two-dimensional Navier-Stokes turbulence has been in evidence over the last several years. Much of the motivation has been meteorological, and stems from the prediction (Kraichnan 1967; Leith 1968; Batchelor 1969) of dual cascades: a simultaneous energy cascade to long wavelengths together with an enstrophy cascade to short wavelengths, both proceeding away from a source of localized excitation in wave number space. Kolmogoroff dimensional arguments lead to an omni-directional energy spectrum $\sim k^{-3}$ in the enstrophy-cascading inertial subrange, above which enstrophy is dissipated by viscosity at the same rate

at which it is being supplied to the fluid. The same Kolmogoroff arguments applied to the inverse energy-cascading inertial subrange lead to an omni-directional energy spectrum $\sim k^{-5/3}$, and an eventual accumulation of energy in the longest wavelengths allowed by the boundary conditions. Considerable meteorological support for this idea exists, and a number of numerical investigations of the viscous-decay initial-value problem have sought the k^{-3} subrange. However, to our knowledge, the only forced two-dimensional turbulence which has been simulated numerically, and for which it has been reasonable to expect an inverse cascade, is in the work of Lilly (1969, 1971).

Lilly's approach to the problem has been followed rather closely in our generalizations to MHD. Lilly's work has recently been criticized by Herring et al. (1974) as spanning an insufficient range in wave number space to draw firm conclusions about values of spectral exponents in the two inertial subranges. We must concede the validity of this criticism. But while there is perhaps too much scatter in the computed spectra to draw firm conclusions about exponents, the existence of two different regions of the spectrum, the lower one of which shows substantial back-transfer of energy in wave number space, seems to be beyond dispute in Lilly's 1969 paper. The two regions are also wholly compatible with the $k^{-5/3}$ and k^{-3} laws. In our opinion, the back transfer is the most physically significant effect in the entire problem. Since the analogous

back-transfer of vector potential is what we most want to study in the MHD case, the inability to draw sharp conclusions about power law spectra seems a reasonable price to pay for its observation. Finally, Lilly's work is at this point still "the only game in town," and extending the range of allowed wave numbers to the point where the criticism of Herring et al. no longer applies would require a greater investment of computing capacity than anyone has as yet been willing to make.

The essence of the criticism may be paraphrased as follows. Suppose the maximum allowed wave number is $k_{\max}$ and the minimum wave number is $k_{\min}$. Suppose the forcing terms added to the right hand side of the Navier-Stokes equation are centered in the wave number space at a forcing wave number k_F . Call the enstrophy dissipation wave number k_v [$k_v \equiv (\eta/v^3)^{1/6}$, where η = the average enstrophy per unit time supplied to the fluid, and v = the kinematic viscosity]. Then in order to have enough space between the various wave number ranges for both inertial subranges to be seen clearly, we require the following chain of inequalities:

$$k_{\min} \ll k_F \ll k_v \ll k_{\max} \quad (1)$$

If each of these inequalities represents at least a factor of 10, then the desired ratio of $k_{\max}/k_{\min}$ is $\geq 10^3$. For Lilly's Navier-Stokes code (1969), this ratio was 32, and for our slower-running MHD code (a single run occupies about an hour of CDC 7600 time), $k_{\max}/k_{\min}$ is only 16. Numerical solutions for MHD turbulence in

which the basic grid has $k_{\max}/k_{\min} \sim 1000$ lie beyond our present capacity. Navier-Stokes computations with $k_{\max}/k_{\min} \geq 1000$ are presently at the threshold of numerical capability (Herring and Orszag, private communication), but no computations have as yet been reported.

It should be kept in mind throughout the following that in the MHD simulations under consideration, with two energy dissipation wave numbers (one constructed from viscosity, one from resistivity, in general unequal), no such chain of inequalities as (1) is possible either. Inevitably, the small size of our $k_{\max}$ will affect the upper end of the calculated spectra in unphysical ways, and will prohibit our observing a proper dissipation range.

In § 2, we describe the general numerical procedure. In § 3, we address first the Navier-Stokes case (Lilly's problem); the bulk of the MHD results are presented in § 4. Some speculations on possible relations between MHD cascades and the "disruptive instability" are offered in § 5. A general discussion of the results appears in § 6.

It is well to stress the differences between these simulations and the earlier ones we have performed for the non-dissipative initial-value problem. Though those are an essential preliminary to these, these are the realistic ones. For the non-dissipative equilibria, there is no energy transfer between different ranges of k at all, and the spectral levels never become independent of the finite $k_{\max}$, for which there is no physical determination. In the presence of forcing and dissipation, energy

transfer between various k values is the most central physical process, and though the system is continually trying to go towards an equilibrium state, it never succeeds. The interesting part of the problem is the balance struck between : (1) the input of excitations; (2) their dissipation; and (3) their thwarted attempt to reach equilibrium.

2. FORCED MAGNETOHYDRODYNAMICS

As in our previous work, we employ periodic boundary conditions in the xy plane. The MHD field variables have the following geometry:

$$\underline{v} = \text{velocity field} = (v_x, v_y, 0)$$

$$\underline{B} = \text{magnetic field} = (B_x, B_y, 0) = \nabla \times \underline{a}$$

$$\underline{a} = \text{vector potential} = (0, 0, a_z)$$

$$\underline{j} = \text{electric current density} = (0, 0, j_z) = \nabla \times \underline{B}$$

$$\underline{\omega} = \text{vorticity density} = (0, 0, \omega_z) = \nabla \times \underline{v}$$

All quantities are expressed in terms of the appropriate dimensionless variables (Fyfe and Montgomery 1975). All quantities are independent of z , but are functions of x , y , and the time t .

Supplementary constraints are that the $\underline{v}$ and $\underline{B}$ fields be solenoidal ($\nabla \cdot \underline{v} = 0$ and $\nabla \cdot \underline{B} = 0$); these constraints are automatic when the equations of motion are written in the current-vorticity (j, ω) representation.

Assuming periodic boundary conditions in x and y , the Fourier-transformed MHD equations reduce to

$$\begin{aligned} \frac{\partial \omega(\underline{k}, t)}{\partial t} = & \sum \underline{M}_1(\underline{r}, \underline{p}) \delta(\underline{p} + \underline{r} - \underline{k}) [\omega(\underline{r})\omega(\underline{p}) - j(\underline{r})j(\underline{p})] \\ & - \nu k^2 \omega(\underline{k}, t) + f(\underline{k}, t) \end{aligned} \quad (2)$$

and

$$\begin{aligned} \frac{\partial j(\underline{k}, t)}{\partial t} = & \sum M_2(\underline{r}, \underline{p}) \delta(\underline{p} + \underline{r} - \underline{k}) [j(\underline{r})\omega(\underline{p}) - \omega(\underline{r})j(\underline{p})] \\ & - \mu k^2 j(\underline{k}, t) + g(\underline{k}, t) \end{aligned} \quad (3)$$

In Eqs. (2) and (3), all quantities have been Fourier-analyzed over a square box, so that, e.g.,

$$\omega_z = \sum_{\underline{k}} \omega(\underline{k}, t) \exp(i\underline{k} \cdot \underline{x})$$

$$\dot{\omega}_z = \sum_{\underline{k}} j(\underline{k}, t) \exp(i\underline{k} \cdot \underline{x}) \quad .$$

The sums are over a large but finite set of $\underline{k}$ -values of the form $\underline{k} = 2\pi (n_x, n_y)/L$ and L is the box dimension. n_x and n_y are integers, not both zero. $\delta(\underline{p} + \underline{r} - \underline{k})$ is a Kronecker delta function of its argument. ν is a dimensionless viscosity and μ is a dimensionless resistivity. μ/ν is sometimes called the "magnetic Prandtl number." ν^{-1} , μ^{-1} are essentially Reynolds numbers, mechanical and magnetic, respectively.

On the right hand sides of (2) and (3), $f(\underline{k}, t)$ and $g(\underline{k}, t)$ are the "forcing terms," mechanical and magnetic, respectively. They are regarded as the sources of the turbulent excitations which are dissipated by the viscosity and resistivity at the large values of k , after the excitations are shuttled through

$\underline{k}$ -space by the nonlinear terms on the right hand sides of (2) and (3). The coupling coefficients M_1 and M_2 are:

$$M_1(\underline{r}, \underline{p}) = \frac{1}{2} \hat{e}_z \cdot (\underline{r} \times \underline{p})(p^{-2} - r^{-2})$$

$$M_2(\underline{r}, \underline{p}) = \frac{1}{2} \hat{e}_z \cdot (\underline{r} \times \underline{p}) |\underline{p} + \underline{r}|^2 p^{-2} r^{-2} \quad (4)$$

The forcing terms $f = f(\underline{k}, t)$ and $g = g(\underline{k}, t)$ are regarded as given functions of $\underline{k}$ and t which model physical processes which may excite the MHD fluid. The range of physical processes which f and g may model is quite wide. For instance, they may be associated with a micro-instability of a non-MHD character which couples to the MHD motions as it grows. Also, f and g could represent stirring of an MHD fluid by a rigid conducting object or boundary. Or, f and g could be the result of an electric current driven in the fluid by an external transformer or capacitor bank. In the case of Navier-Stokes fluids, f can represent such different sources as a screen in a wind tunnel or the "baroclinic instability" (Pedlosky 1971) in meteorology.

In any particular application, the functional form of f and g can be obviously quite complicated. However, in the spirit of homogeneous turbulence theory, we are seeking physical processes which are not specific to a unique situation but which are common to many different f 's and g 's. It is in this spirit that Lilly (1969)

gave a recipe for providing an f as a random function of time that is relatively non-committal, and we follow his example for the MHD case.

$f(\underline{k}, t)$ and $g(\underline{k}, t)$ are set identically zero outside a fixed circular annulus ("forcing band") in k_x, k_y space. Let $H_{\underline{k}}$ be the real or imaginary part of f or g for one of the allowed values of $\underline{k}$ in this band. The $H_{\underline{k}}$ at time step $n + 1$ is related to $H_{\underline{k}}$ at time step n by

$$H_{\underline{k}}(n + 1) = R H_{\underline{k}}(n) + \sqrt{1 - R^2} J_{\underline{k}}(n) \quad , \quad (5)$$

where R is a fixed real number between 0 and 1, and $J_{\underline{k}}(n)$ is chosen by a Gaussian random number generator with expectation value zero and variance such that the mean $\langle H_{\underline{k}}^2 \rangle$ remains constant. In other words, the Fourier components are composed of a part which is a fraction of their value at the previous time step plus a random part chosen so that the expectation value of the magnitude of the forcing term is a constant. The evolution of the forcing field can be graphically portrayed as random walk over the surface of a hypersphere in the space of Fourier coefficients. The forcing term is programmed so that f and g can be chosen to be separately zero, to be uncorrelated, or to be identical. The limit $R = 1$ corresponds to a temporally constant forcing field, and the limit $R = 0$ to a

wholly random (and highly discontinuous) white noise field. The typical running value of R has been 0.95.

The rest of the numerical scheme is very similar to that used by Fyfe, Joyce, and Montgomery (1976). It is a spectral-method code utilizing the Fast Fourier Transform, along the lines of Orszag (1971) and Patterson and Orszag (1971). The main accuracy checks have been tests of the conservation of the three non-dissipative invariants (Fyfe et al. 1976) with the dissipation and forcing terms turned off. To gain additional confidence in the program, we have chosen first to re-investigate the pure Navier-Stokes case [obtained by setting $j \equiv 0$ and $g \equiv 0$ in (2) and (3)] in order to compare our results with those of Lilly (1969). This work is described in § 3.

A final remark is in order concerning the way the numerical integration is carried out. Though the symmetry of Eqs. (2) and (3) renders them easiest to manipulate from a formal point of view, numerical calculations turn out to be easiest to perform on the equivalent pair of equations (Fourier-transformed)

$$\frac{\partial \omega_z}{\partial t} = -\tilde{v} \cdot \nabla \omega_z + \tilde{u} \cdot \nabla j_z + \nu \nabla^2 \omega_z + F \quad (6)$$

$$\frac{\partial a_z}{\partial t} = -\tilde{v} \cdot \nabla a_z + \mu \nabla^2 a_z + G \quad , \quad (7)$$

where F , G are now the random forcing functions. Because the equation for a_z , rather than j_z , is used, the Fourier transform of G is not $g(k, t)$, but rather $g(k, t)/k^2$. The implication is that if the forcing terms are to supply kinetic and magnetic energy at approximately the same rate, F and G are not to be comparable in magnitude, but rather the ratio of the Fourier transform of G to the Fourier transform of F must be of the order of the reciprocal of the square of the forcing wave number k_F . Unless specifically stated otherwise, it may be assumed in what follows that this has been taken into account and that when f and g are both non-zero, they are of the same order of magnitude.

The typical magnitude chosen for the forcing field is $\Sigma \langle |f(k, t)|^2 \rangle = 16$, where the summation is over those values of k^2 between 55 and 70. A comparable value of $\Sigma \langle |g(k, t)|^2 \rangle$ is chosen when magnetic forcing is present. Experiments have also been carried out with $\Sigma \langle |f(k, t)|^2 \rangle = 64.0$ and 4.0 ; the net effect seems to be to accelerate or decelerate the processes, and to raise or lower the limiting modal energies, but not to alter them qualitatively.

3. FORCED NAVIER-STOKES TURBULENCE IN TWO DIMENSIONS

Three innovations not employed by Lilly (1969) have made it possible to improve some details of his computation. (1) We use the Orszag-Patterson spectral method, not yet operative in 1969. (2) We average modal energies over all $\underline{k}$ -values corresponding to the same k^2 , instead of averaging around the perimeters of squares in $\underline{k}$ -space, as Lilly did. (3) We time average the modal energies once the total enstrophy has become approximately constant and once the rate of absorption of total energy has become approximately constant. All three improvements may reasonably be expected to reduce the errors in the spectrum slightly.

Results for a typical Navier-Stokes run are shown in Figs. 1 and 2. These refer to a forced, purely Navier-Stokes run, starting with an empty spectrum. This run, as all others, is carried out on a 32×32 grid with $k_{\min}^2 = 1$, $k_{\max}^2 = 226$, and a box size $L = 2\pi$. These parameters characterize all runs. The time step Δt is $(128)^{-1}$, and this value characterizes all runs unless it is explicitly stated to the contrary. Some runs were carried out with $\Delta t = (256)^{-1}$, and there are some explicit comparisons described later for the MHD case. Fig. 1 shows the total computed enstrophy $\Omega = \frac{1}{2} \sum_{\underline{k}} |\omega(\underline{k}, t)|^2$ and computed energy $\epsilon = \frac{1}{2} \sum_{\underline{k}} |\omega(\underline{k}, t)|^2 k^{-2} = \frac{1}{2} \sum_{\underline{k}} |\chi(\underline{k}, t)|^2$ versus time.

The behavior of energy would be a linear increase with time and the behavior of enstrophy would be an approach to a constant, if the Kraichnan conjecture (1967) were given its most straightforward interpretation.

The forcing wave number is approximately $k_F = 8$. More specifically, the forcing wave numbers are confined to a band extending from $k^2 = 55$ to $k^2 = 70$. Figs. 2a, 2b are time averaged modal energy plots. Figure 2a shows the time average of $|\underline{v}(\underline{k}, t)|^2$, averaged for given values of k^2 , for 320 time steps ending at $t = 37.5$. Every value of k^2 is plotted for $1 \leq k^2 \leq 30$, every other point is plotted for $30 \leq k^2 \leq 100$, and every fourth point is plotted for $k^2 \geq 100$. The allowed values of k^2 get very dense at the higher values, and the above plotting convention is followed throughout the paper in order to keep the high- k modal energy plots from becoming unmanageably cluttered. Fig. 2b is a similar modal energy plot for an average of 1200 time steps ending with $t = 75.0$. The straight lines drawn on the graph above and below the forcing wave numbers have slopes -4 and $-8/3$, respectively, corresponding to omni-directional energy spectra $2\pi k \langle |\underline{v}(\underline{k})|^2 \rangle \sim k^{-3}$ and $k^{-5/3}$. (Since the locations of the k_x, k_y points are far from isotropic at the lowest values of k , it seems misleading to convert the modal energy plots into plots of omni-directional spectra.) Figure 2c is a plot of $k^4 |\underline{v}(\underline{k})|^2$ vs. k^2 , time averaged. Theoretically, this quantity should be flat in the enstrophy-cascading

subrange, and be proportional to $k^{4/3}$ in the energy-cascading subrange. The straight lines have these slopes.

Though it is apparent that the results are not inconsistent with the -3 and $-5/3$ predictions, the data do not permit the inference that these are the values. A least-squares fit of the expression Ak^{-n} to $k\langle|v(k)|^2\rangle$ for the data gives $n \approx 1.49$ for k^2 below 55 and $n \approx 2.07$ for $k^2 \geq 70$. It has not been possible to observe the predicted build-up of the $k^2 = 1$ excitations above the predicted level obtained from the $k^{-5/3}$ law. Though at the end of the run, the $k^2 = 1$ modes are still growing, the 9500 time steps represented in Figs. 1 and 2 are already about 50% beyond the time over which the code will conserve energy and enstrophy below the 5% level for the inviscid unforced case. We believe that running longer would produce an arbitrarily large $k^2 = 1$ build-up, but the accuracy of the solution would have been lost.

The enstrophy dissipation can be estimated as $\eta \approx 2\nu k_F^2 \approx k_F^2 \times [\text{the energy dissipation}] \approx 2.4$, for the last 1200 time steps. This gives a dissipation wave number $k_v \approx (2.4/10^{-9})^{1/6} \approx 40$, actually above $k_{\max}$. This makes the consistency of all but the highest- k points of Fig. 2b with the $k^{-5/3}$, k^{-3} prediction quite remarkable, and suggests that perhaps the inequalities (1) are too stringent.

In looking at modal energy plots such as Fig. 2, one should not be misled, by the logarithmic shrinkage of k space and by the

plotting conventions, into underestimating the number of modes in the upper reaches of the wave number space. For example, there are almost four times as many different $\underline{k}$ modes above $k_F = 8$ as there are below it. We have chosen to plot k^2 logarithmically, because in this particular application we are most interested in the back transfer to the lower values of k . It is the small- k part of the Fourier space that is emphasized by the log-log plots.

One should also keep in mind that each value of k^2 has a multiplicity of 2 to 8. That is, for each k^2 , there are 2 to 8 independent $\underline{k}$ values, and these degenerate $\underline{k}$ modes are always averaged over.

4. FORCED MHD TURBULENCE IN TWO DIMENSIONS

It is well to remark at the outset the main qualitative difference that has been consistently observed between the MHD runs and the Navier-Stokes runs illustrated in § 3. This is that the MHD case seems to be unavoidably "noisier" than the Navier-Stokes case. As already noted (Fyfe, Joyce, and Montgomery 1976) for the non-dissipative initial value problem, there is a slowing-down of magnetic energy transfer among the lower k 's which makes time averaging less effective at smoothing out spectral fluctuations. Most of the MHD spectra are less smooth than those shown in Figs. 2.

A. The Case $g = 0$ (Mechanical Forcing)

The first case we consider is the case $g(\underline{k}, t) \equiv 0$. There is no significant amount of back transfer of vector potential for this case. The mean square vector potential can only decay. It is uncertain whether there is any physically realistic process which is satisfactorily modelled by setting $g(\underline{k}, t) \equiv 0$. For instance, mechanically stirring the fluid by constraining the velocity field to have a specified value at a solid surface also constrains the magnetic field at the same boundary (particularly if the solid has a finite conductivity) and so cannot be properly modelled without a magnetic forcing term. Nevertheless, we cannot entirely rule out

processes for which $g \equiv 0$, and we now present a few results for that case.

Fig. 3 shows the enstrophy and energy vs. time for a run with $g(k, t) \equiv 0$, $\nu = \mu = 10^{-3}$, with a small initial "seed" magnetic field. Without magnetic field amplification, one might expect this case to yield something close to Navier-Stokes behavior, and it does. Energy increases linearly with time and enstrophy fluctuates about a constant value. The corresponding modal energy plot (time averaged over 1200 time steps ending at $t = 75.0$) is shown in Fig. 4; this is to be compared with Fig. 2b.

One interesting and not yet quantitatively explained feature of this case is shown in Fig. 5, a magnetic modal energy spectrum averaged over 1200 time steps ending at $t = 75.0$. (This spectrum was flat, with all values $= 10^{-6}$ at $t = 0$. From $t = 0.0$ to $t = 75.0$, the magnetic energy has decayed by a factor of 2×10^5 .) This steeply rising character for the very low level of initial magnetic excitations has appeared in all the purely mechanically forced runs we have done, including some with still smaller seed fields. (The final ratio of magnetic to mechanical energy for this run is $\sim 10^{-9}$, so that the errors in the computation of the velocity field are surely larger than the sizes of the magnetic components; but this is no reason to doubt the accuracy of the magnetic field calculation, assuming the velocity field to be given.)

A simple argument predicts qualitatively the growing $|B(k)|^2$ vs. k^2 behavior. Suppose we regard the magnetic field as so

weak that it does not affect the velocity field $\underline{v}(\underline{x}, t)$. Then the equation for the vector potential (omitting dissipation)

$$\frac{\partial \underline{a}_z}{\partial t} + \underline{v} \cdot \nabla \underline{a}_z = 0 \quad ,$$

shows that $\underline{a}_z$ is effectively convected like a passive scalar in a given turbulent velocity field, the effect of which will be to scramble the $\underline{a}_z$ field into progressively higher k modes. Batchelor (1959) has argued that under some circumstances, the omni-directional spectrum for such a passive scalar should approach $\sim k^{-1}$ behavior. This would imply an omni-directional magnetic field spectrum $\sim k^{+1}$. The actual rates of increase in such curves as Fig. 5 are typically higher than that, increasing more like $k^{2.5}$. This discussion ignores the Alfvén effect (discussed in Appendix A), which would predict a more rapid rise with k than that for a passively-convected $\underline{a}_z$. But we have no quantitative argument for the $k^{2.5}$ behavior shown in Fig. 5. The relation of the non-dissipative MHD and Navier-Stokes cases is discussed in Appendix B.

B. The Case $f = 0$ (Magnetic Forcing)

The next case to be considered is the case of wholly magnetic forcing, $f \equiv 0$. We are again uncertain as to the degree of realism to be assigned to such a model, but for the sake of completeness, we discuss it before passing on to the case where neither f nor g vanishes. Again, $\nu = \mu = 10^{-3}$.

We begin from a completely empty spectrum

$$(|\underline{B}(\underline{k})|^2 = 0, |\underline{v}(\underline{k})|^2 = 0) \quad .$$

Fig. 6a shows the instantaneous magnetic modal energies at $t = 75.0$. Fig. 6a reveals considerable scatter in the spectrum, but also significant back-transfer of magnetic energy to the longest wave length modes (recall that the forcing wave numbers are from $k^2 = 55$ to $k^2 = 70$). Fig. 6b shows the time-averaged (1200 time steps ending at $t = 75.0$) magnetic modal energies for the same run. The time averaging eliminates some of the fluctuations, but not as much as for the Navier-Stokes case. The broken line has slope $k^{-2/3}$, corresponding to an omni-directional energy spectrum $\sim k^{-3}$, though in no sense is it implied that the slope can be inferred from the points. The corresponding kinetic modal energies are shown in Fig. 6c, time averaged over 1200 time steps ending at $t = 75.0$. Fig. 7 is a plot of total energy and vector potential vs. time for this run, and the cross-helicity P is shown also. As can easily be seen, vector potential is added at a constant rate, while the energy reaches a limiting value rather quickly and executes 10% fluctuations about that value. This is essentially what one expects for the dual-cascade behavior, and Fig. 7 should be compared with Fig. 3. The cross-helicity is not identically zero, and fluctuates about zero with an amplitude of the order of .01. The

ratio of magnetic energy to kinetic energy at $t = 75.0$ is 1.74, so that the conversion of magnetic into mechanical energy has been substantial.

By noting the rate of addition of squared vector potential ($dA/dt \approx 5 \times 10^{-4}$ from Fig. 7a) we may estimate the rate of addition of magnetic energy, which, since the energy added is only magnetic, enables us to estimate the rate of addition of energy,

$$\dot{\epsilon}(t) \approx k_F^2 dA/dt \approx .032.$$

From this we may estimate the dissipation wave number

$$k_v = k_\mu = [\dot{\epsilon}(t)/\nu^3]^{1/4} \approx (.32)^{1/4} \times 10^2 \approx 75,$$

above $k_{\max}$.

Another feature to be noted in Fig. 6a and Fig. 6b is the relative enhancement of the spectrum in the forcing band; the dissipation of energy away from the forcing wave numbers is somewhat greater than it is for the Navier-Stokes case.

A tentative explanation of this last-named effect is the following. The forcing wave numbers lie within a relatively narrow band, and the first step in the excitation of the turbulence is the direct excitation of these wave numbers with $k \approx k_F$ by the forcing terms. The next step is the transfer of energy out of the forcing

band by the nonlinear interactions measured by the coupling coefficients M_1, M_2 [Eq. (4)]. Now the ratio of the magnetic coupling coefficient to mechanical coupling coefficient for the interaction of two wave numbers r, p is

$$\frac{M_2(r, p)}{M_1(r, p)} = \frac{|p + r|^2}{p^2 - r^2}.$$

For p, r about the same size, this ratio is in general quite large. Thus for a narrowly forced band of excitations, magnetic excitations get out into the main body of the distribution faster than mechanical excitations. Since the excitations arriving at the higher values of k are dissipated first, the overall dissipation is expected to be higher for the case $f = 0$ than it is for the case $g = 0$.

C. Mechanical and Magnetic Forcing ($f \neq 0, g \neq 0$)

One surprise has occurred repeatedly, in the approximately 12 runs (9600 time steps each) we have made for which we have used simultaneous magnetic and mechanical forcing: namely, a nearly flat spectrum of magnetic modal energies at all except the lowest values of k . This effect has persisted for different values of ν, μ ; for correlated and uncorrelated f and g ; for variations in the magnitude of f and g ; and for variations in the size of the time step. The effect is so pronounced as to be identifiable as a

qualitative difference from the Navier-Stokes results (Figs. 2a, 2b, 2c), from the mechanically forced MHD case (Figs. 4, 5), and from the magnetically forced MHD case (Figs. 6a, b, c). We are not confident that we have a correct explanation for the phenomenon, but we attempt one at the end of this section.

Figs. 8 and 9 are not dissimilar to the cases previously discussed. Figs. 8 and 9 refer to the case of equal and uncorrelated f and g , for $\mu = \nu = 10^{-5}$, and $\Delta t = (128)^{-1}$. The forcing wave numbers are again $k^2 = 55$ to 70 . Fig. 8 is a plot of total energy, total cross helicity, and total mean square vector potential vs. time. It shows the energy ϵ saturating and fluctuating about a saturated value, the cross helicity P executing small fluctuations about zero, and the mean square vector potential A increasing monotonically with time, much as in Figs. 7a, 7b. Fig. 9 shows the mechanical energy, the magnetic energy and their ratio $R \equiv [\text{magnetic energy/kinetic energy}]$ vs. time. Both energies saturate, and their ratio fluctuates about a value slightly larger than unity.

This behavior is so far consistent with the picture of the dual vector potential and total energy cascade we have been proposing. But a considerably more complex situation presents itself when we turn to the spectral plots. Figs. 10a, 10b are time averaged modal energies vs. k^2 for the magnetic field and velocity field. Substantial dynamo action (back transfer of magnetic energy) has

obviously occurred, but except for the lowest magnetic modes, there is little structure of any kind, and comparisons with spectral indices cannot mean very much.

Figs. 11a, b show exactly the same run, with the difference that now $\nu = \mu = 10^{-2}$. Increasing the dissipation by an order of magnitude is seen to have selectively dissipated the velocity field (there is no tendency for back transfer for $|\underline{v}(\underline{k})|^2$), and to have substantially inhibited the back transfer of magnetic energy. The dynamo effect can, not surprisingly, be wiped out by making the viscosity and resistivity high enough. Neither ϵ nor A show any systematic growth, for the situation depicted in Figs. 11, after about $t = 3.0$.

In Figs. (11), the modal energies for the forcing wave numbers are off scale, at the level of $\gtrsim 1.4 \times 10^{-3}$. This implies that by far the strongest nonlinear interaction is among the forcing band wave numbers, and these give the largest single contribution to the flux of energy into the high- k parts of the spectrum. We have previously noted that the ratio of the magnetic coupling coefficient to the mechanical coupling coefficient,

$$\frac{M_2(\underline{r}, \underline{r})}{M_1(\underline{r}, \underline{r})} = \frac{(\underline{r} + \underline{r})^2}{\underline{r}^2 - \underline{r}^2}$$

is in general very large for similar-magnitude wave vectors $\underline{p}, \underline{r}$.

This implies that the highly excited forcing band modes in Fig. 11 are more effective at driving magnetic excitations outside the forcing band than mechanical ones. The effect is strong enough to overcome the "Alfvén effect" (Appendix A) and separate the magnetic and mechanical spectra as shown at the upper k values in Fig. 11.

We have also done runs with uncorrelated f , g of equal magnitude and transport coefficients μ , $\nu = 10^{-2}$, 10^{-3} and 10^{-3} , 10^{-2} . Perhaps unsurprisingly, the spectrum associated with the larger transport coefficient was depressed relative to the other, even at high k , by as much as an order of magnitude. By varying the μ/ν ratio, the excitations can be made dominantly mechanical or dominantly magnetic.

Examples of the kind of behavior exhibited in Figs. 8, 9, 10 could be multiplied, but it seems redundant here to do so. We have a partial explanation to suggest for the behavior, though its confirmation will await simulations capable of considerably higher spatial resolution than $k_{\max}/k_{\min} = 16$. We may estimate the "decay time" for wave number k by $\tau_D \equiv (k^2 \nu)^{-1}$ or $(k^2 \mu)^{-1}$, whichever is shorter. τ_D at the forcing wave number is about 15, and at the maximum wave number ($k_{\max}^2 = 226$), is about 5. We do not have an accurate expression for the rate of transfer of energy among different k values, but it may be faster than the eddy turnover time, $\tau_k \equiv (k|\chi(k)|)^{-1}$, in any given region of k space. From Fig. 10, τ_k is about 4 for the forcing wave numbers and about 2.5

for $k_{\max}$. We have the situation, then, that $\tau_k \leq \tau_D$ for all k between k_F and $k_{\max}$. τ_k may reasonably be expected to measure the time in which the non-dissipative equilibria are locally attained (Fyfe, Joyce, and Montgomery 1976). The fact that the energy transfer among the allowed wave numbers may occur more rapidly than the maximum time of dissipation of any wave number could imply that for these simulations, the thermal equilibrium spectra (Fyfe and Montgomery 1975) are more relevant than the cascade spectra. What is being envisioned is a steady-state process in which energy is being added at a rate which is the same as the rate at which it is being dissipated, but that it rests in what is near to a quasi-stationary thermal equilibrium state. This picture is somewhat substantiated by the dissipation spectra ($|\omega(k)|^2$, $|j(k)|^2$) plotted in Fig. 12. These are seen not to have the characteristic behavior expected for cascade processes in their classical Kolmogoroff-Kraichnan form: a maximum, then decay near the upper end of the appreciably excited wave numbers.

What the foregoing argument fails to explain are the differences in the MHD and Navier-Stokes dynamics. At the same rate of injection of energy, the Navier-Stokes fluid with identical transport coefficients is simply more efficient at disposing of the excitations at high k than the MHD fluid. The only explanation for this effect that we have been able to arrive at is the already noted more rapid high- k magnetic energy transfer. If correct

the implication of this for the accurate numerical simulation of full-blown dual cascades in MHD is somewhat depressing: namely, even a larger $k_{\max}/k_{\min}$ is required than for the Navier-Stokes case. The current value of attainable $k_{\max}/k_{\min}$ for the Navier-Stokes case of 1024 (Herring and Orszag, private communication) is already on the threshold of present computational capacity. The MHD case may await the advent of the fifth-generation machines before it can be exhaustively studied. The more aware reader will have perceived that these considerations raise some questions about the adequacy of the spatial resolution of very nearly every MHD simulation that has as yet been performed.

We close this section with some remarks on the effects of doubling the magnitudes of the forcing terms f, g . Figs. 13 show modal energies for f, g twice as large as in Figs. 10. There is little change in the overall character of the plots, except for the enhancement of the $k^2 = 1$ magnetic excitation by a significant amount. This is the strongest "dynamo effect" we have observed in any single run.

Other runs were carried out for the same parameters as Figs. 10 involve, but with a time step $\Delta t = (256)^{-1}$. No qualitative differences were observed in the spectra, though of course the spectra will not be identical since in general, different sets of random numbers will be involved in the forcing terms.

5. EXERCISE: A TURBULENT MHD CASCADE MODEL OF A "DISRUPTIVE INSTABILITY"

Under conditions of too high current or number density, tokamak discharges exhibit a behavior that has acquired the name "disruptive instability." (For an encyclopaedic survey of the subject up to 1974, see Furth 1974; for recent experimental results, see, e.g., Hutchinson 1976 or Morton 1976; for recent theoretical approaches, see, e.g., Finn 1975 or Stix 1976). Essentially what happens is that the current channel in a cylindrical plasma column suddenly expands to fill the channel in a highly disordered way, in effect terminating the confinement (we ignore the toroidal aspects of the problem in this qualitative discussion).

We wish to suggest here, without the extraordinary numerical effort that would be required in order to follow through on the details, a model of this behavior which starts from a turbulent MHD cascade framework. Previously the phenomenon has been approached from a non-statistical MHD framework, usually (and not wholly successfully) from the point of view of linear MHD stability. The computational effort required to implement the program now to be outlined is not to be underestimated; numerical techniques would be required that cannot be described as "off the shelf" at present.

For computational convenience, virtually all the basic theoretical and computational studies of either Navier-Stokes or MHD turbulence have been carried out in rectangular geometry. The turbulent fields have been expanded, assuming periodic boundary conditions, in the appropriate orthogonal functions, $\exp i \underline{k} \cdot \underline{x}$. Though expanding in some other set of eigenfunctions of the Laplacian would have increased the computational labor considerably, there is nothing fundamental about the restriction to rectangular geometry. For example, had we been interested in doing the present MHD calculation in the presence of rigid, perfectly-conducting circular boundaries instead of rectangular boundaries, an appropriate set of orthogonal functions in which to expand j_z and ω_z might have been $J_m(k_n r) \exp(im\varphi)$, instead of $\exp i \underline{k} \cdot \underline{x}$ [here (r, φ) are polar coordinates, m is any integer, and $k_n a$ is the n th zero or node of the Bessel function J_m]. In fact, such a set of orthogonal functions would have had the advantage of permitting a non-zero value of the total current, $\iint dx dy j_z$, which in the case of periodic boundary conditions on the vector potential is constrained to be zero. (A non-zero value of this integral is a central component of tokamak confinement.) The disadvantages would have been the loss of the utility of the fast Fourier transform and the Orszag-Patterson spectral method: both severe losses.

If the MHD equations, in the presence of other boundary conditions, are expanded in terms of another set of eigenfunctions,

their formal structure may be represented as

$$\frac{dX_i}{dt} = \sum_{jk} C_{ijk} X_j X_k - \nu_i X_i + F_i \quad (8)$$

Eq. (8) specialized to rectangular geometry and periodic boundary conditions, is just Eqs. (2) and (3). X_i is the i th element of a (possibly very long) column vector composed of all the independent expansion coefficients of all the necessary MHD variables. [In the problem considered in this paper, X_i represents symbolically all the $w(\underline{k})$ and $j(\underline{k})$ for the allowed range of $\underline{k}$.] The C_{ijk} are the coupling coefficients, given by Eqs. (4) for rectangular geometry; but in more complicated cases they would probably require numerical evaluation and storage. The F_i is an external forcing term (possibly random, possibly not). F_i might represent, e.g., the electric field of the external transformer in the tokamak discharge; in that case, all but a few of the F_i would vanish. Finally, ν_i is the viscous or resistive decay coefficient, the negative of the viscosity or resistivity times the corresponding eigenvalue of the Laplacian, in the present case.

On the basis of the formal system (8), a scenario can be outlined which would lead to the qualitative behavior of the "disruptive instability." (A not totally dissimilar scenario can be imagined for the onset of turbulent flow of water down a pipe.) Suppose that the electric field F_i is confined to one of the

low-lying values of i , say i_0 . By low-lying, we mean, corresponding to one of the smallest values of v_i . A solution of Eqs. (8) is then

$$F_{i_0} = v_{i_0} X_{i_0} \quad (9)$$

with all the other $X_i \approx 0$. In reality, the v_i will have a temperature dependence, and are expected to be gradually reduced by the ongoing dissipation implied by Eq. (9). The bigger F_{i_0} is, the faster will be the reduction.

We have seen in the preceding pages that a turbulent cascade can be efficiently inhibited by increasing the dissipation rates. It may reasonably be inferred that a cascade may be launched, for a given forcing term F_{i_0} , when the v_i fall below critical values. (Antecedents of this idea appear in Kadomtsev 1975 and Sykes and Wesson 1976.)

For $i \neq i_0$, the initial stage of the developing turbulent cascade might be described by discarding all the terms in the sum in Eq. (8) except those involving X_{i_0} . X_{i_0} and v_{i_0} could be considered to a first approximation to be stationary in time. The growth of the other X_i (in this phase and this phase only) would appear as a linear instability, for which the existence of a positive growth rate would involve a threshold condition involving F_{i_0} , X_{i_0} , and the v_i . As soon as the other X_i had grown out of the linear range, all the terms in Eq. (8) would have to be considered, and a full-blown turbulent cascade would be under way. Its accurate numerical description would be subject to all the stringent

requirements on spatial resolution that our present simulation has been afflicted with, and without the short-cuts afforded by the fast Fourier transform and the Orszag spectral method. It is no exaggeration to say that no MHD simulation for any problem has yet been performed with spatial resolution small enough to avoid serious unphysical grid effects in a turbulent cascade. Whether this program, or its even more demanding three-dimensional analogue, could or should be carried out requires consideration of issues beyond the scope of this article.

However, it is apparent that the above scenario does lead to a qualitatively correct behavior: excitation of the X_1 corresponding to coefficients of Bessel functions of higher index will manifest itself as a spreading of the j_z distribution over the whole cross-section. The enhanced transport that would surely result from the excitation of the vorticity coefficients would imply a transport of the MHD fluid to the walls at the same time. No obvious solution to the problem posed by the "disruptive instability" is suggested by the above considerations. The way to get rid of a turbulent MHD cascade is to make the plasma more resistive: cool it off or add more impurities. Neither message is particularly exhilarating in the context of controlled fusion research.

Finally we remark that a somewhat more strained interpretation of the "disruptive instability" as an inverse cascade can be given. If the current distribution prior to the onset of the disruptive

instability is not as smoothly centered as has been assumed, but rather is either closely confined to the center or is filamented (and the spatial resolution of the measurements of Hutchinson 1976, for example, do not seem to preclude this possibility, though they do not suggest it either), then the source of the excitations might be far out in k -space (or m, n space). Then when a certain threshold was crossed, an inverse vector potential cascade to long wavelengths might result, or an inverse cascade of magnetic helicity for three-dimensional excitations (e.g., Pouquet et. al 1975).

6. DISCUSSION

It is considered that in the previous pages, it has been numerically established that dynamo action, in the form of substantial back-transfer of vector potential, can occur for a two-dimensional magnetohydrodynamic flow with mechanical and magnetic Reynolds numbers large comparable to unity. Present computer capacity does not permit simulations on a large enough grid to be able to draw sharp conclusions about conjectured exponential power laws. The presence of external agencies, microscopic or macroscopic, has been modelled by the presence of forcing terms on the right hand sides of the equations of motion for the velocity field and magnetic field. It has been concluded that magnetic forcing terms must be present for the dynamo action to occur. Most processes which one might imagine as a source of turbulent excitations in an MHD fluid (such as friction with a solid surface of finite conductivity) seem best modelled by a forcing function involving both magnetic and mechanical components.

The translation of the above results into an experimentally testable format may be a slow and painful process. First of all, the diagnostic techniques for measuring fluctuating magnetic fields and velocity fields in even laboratory plasmas are primitive compared to those which have evolved for diagnosing Navier-Stokes

fluids (e.g., Grant, Stewart, and Moilliet 1962; Frenkiel and Klebanoff 1967; Wiin-Nielsen 1967). For example, at present it appears to be difficult to measure the magnetic field interior to a tokamak discharge at all, let alone resolve it spatially over several orders of magnitude. Secondly, there are formidable psychological obstacles to the incorporation into plasma physics of the insights and techniques that have been achieved in the last fifteen years of fluid turbulence theory. "Plasma turbulence," as a subject, has evolved with a great deal of emphasis on linear instabilities and dispersion relations for normal modes of oscillation which are supposed to be a central component in the description of the turbulent field. None of these concepts plays any significant role in the modern theory of fully developed fluid turbulence. Readjusting the emphasis away from the preoccupation with linear apparatus (a twenty-five year enterprise that, in our opinion, has produced much of debatable scientific worth) may be slow indeed. Few industrial fluids, with the exception of those under storage in tanks or reservoirs, are stable in the sense which has been intended by plasma physicists; virtually all are turbulent. Fortunately, it is not necessary to have a good basic understanding of the turbulent flows involved in order to design fluid machinery. A proper and respectful distance has developed between the basic and engineering aspects of fluid mechanics. When a similar division of plasma physics is perceived to be natural and desirable, the subject may be approaching maturity.

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APPENDIX A: THE ALFVÉN EFFECT

The Alfvén effect was first formulated by Kraichnan (1965) and has been invoked recently by Pouquet et al. (1975) in eddy-damped quasi-normal model computations for three dimensional MHD turbulent flows with helicity. The effect can be described more simply than has been done previously, and the description is relevant to some of the other effects we have observed.

The effect concerns the very high- k components of a turbulent MHD field. The lower- k components are approximated by spatially-uniform velocity field and magnetic field $\underline{v}_0, \underline{B}_0$. The higher- k $\underline{v}(k), \underline{B}(k)$ are treated as small-amplitude perturbations on this spatially-uniform state. All possible high- k fluctuations can then be shown to be a linear superposition of normal modes for which the time-averaged values of $|\underline{v}(k)|^2$ and $|\underline{B}(k)|^2$ are equal. In this approximation, one should see $\langle |\underline{v}(k)|^2 \rangle = \langle |\underline{B}(k)|^2 \rangle$ at high k , or equipartition of magnetic and kinetic energy. This has in fact been a prediction of the inviscid, perfectly-conducting equilibrium theory (Fyfe and Montgomery 1975), and has been numerically verified for that case (Fyfe, Joyce, and Montgomery 1976). The equipartition at high wave numbers is a feature of Figs. 6, 10, and 13 of the present paper, and the fluid appears to be striving for the equipartition, even though it never achieves

it, in Fig. 5. However, we can see in Fig. 11 that the effect is not absolute, and the symmetry can be broken by increasing the dissipation, symmetrically or unsymmetrically, between the $\underline{B}$ and $\underline{v}$ fields.

A second conclusion has been drawn about the Alfvén effect: that the linear de-coupling of the fields $\underline{v} + \underline{B}$ and $\underline{v} - \underline{B}$ in the above approximation inhibits high- k energy transfer between wave numbers and thus flattens the energy cascade (an omni-directional spectrum $k^{-\frac{3}{2}}$ is supposed to replace the $k^{-\frac{5}{3}}$ behavior). About this conclusion we are less sure. A transfer between the fields $\underline{v} + \underline{B}$ and $\underline{v} - \underline{B}$ may become more difficult, but a general uncorrelated non-helical forcing term supplies both fields at an approximately equal rate, and there seems to be no reason to suspect a decrease in the rate of transfer among different k 's for either field. Some of the present evidence favors an enhanced magnetic transfer. The conjecture must be left unresolved at present. It should be remarked that in such model calculations as the eddy-damped quasi-normal model (Pouquet et al., 1975), a range of exponents can be achieved by the choice of damping rate for the three-coefficient auto-correlations. The question of the implications of the Alfvén effect for spectral laws must be decided on physical grounds outside the framework of models, since this choice is essentially arbitrary.

APPENDIX B: EQUILIBRIUM THEORY AT LOW A

It was mentioned in Fyfe and Montgomery (1975) that the limit of the inviscid equilibrium theory as the magnetic excitation level approached zero ($A \rightarrow 0$) was different from the inviscid Navier-Stokes equilibrium theory, even though the dynamical equations approach each other in that limit. Here we digress from the main thrust of this paper to describe a numerical experiment which dramatically illustrates this subtlety involved in the limits.

The experiment consists of finding the long-time modal energies for the velocity field for two inviscid, zero-resistivity initial-value problems. The velocity-field coefficients are the same initially for the two runs. The only difference in the initial values for the two runs is the low level initial magnetic excitations in the second run; in the first run they are identically zero.

For both runs, the initially non-vanishing velocity coefficients are for $\underline{k} = (3, 1), (3, 2), (2, 3), (3, -1), (3, -2), (2, -3), (1, 3), (1, -3)$. The initially non-vanishing vector potential coefficients for the second run correspond to $\underline{k} = (2, 2), (3, 0), (0, 3), (2, -2)$. The initial kinetic energy is $\frac{1}{2} \sum_{\underline{k}} |\underline{v}(\underline{k})|^2 = 2.94$ for both runs, and the initial magnetic field energy for the second run is $\frac{1}{2} \sum_{\underline{k}} |\underline{B}(\underline{k})|^2 = 0.085$, 2.89% of the kinetic energy. The prediction

from the 1967 Kraichnan theory for the modal energy spectrum for the first velocity field is $\langle |\underline{v}(\underline{k})|^2 \rangle = (\alpha + \beta k^2)^{-1}$ with $\alpha = -10.22$ and $\beta = 11.17$. The equilibrium theory for the second case predicts $\langle |\underline{v}(\underline{k})|^2 \rangle = \alpha^{-1}$, with $\alpha = 166.5$.

The numerically measured time averaged spectra at the ends of the runs are shown in Figs. 14a and 14 b. The theoretical curves are the solid lines. It is to us extraordinary that this small addition of magnetic energy can, by destroying the constancy of the enstrophy, so alter the qualitative behavior of the velocity field. Indications are that any magnetic excitation, however small, can make this qualitative modification given sufficient time.

FIGURE CAPTIONS

- Fig. 1 Enstrophy $\Omega(t)$ and energy $\epsilon(t)$ vs. time for pure Navier-Stokes case. A saturated constant $\Omega(t)$ and a linearly increasing $\epsilon(t)$ would characterize a perfect dual cascade. $\nu = 10^{-3}$.
- Fig. 2 Modal energies, $|\underline{v}(\underline{k})|^2$ vs. k^2 , corresponding to the Navier-Stokes run shown in Fig. 1. Fig. 2a shows the average modal kinetic energies, averaged over 320 time steps ending with $t = 37.5$. Fig. 2b shows modal kinetic energies averaged over 1200 time steps ($\Delta t = (128)^{-1}$) ending at $t = 75.0$. The forcing band is indicated by arrows drawn at $k^2 = 55$ and $k^2 = 70$. The slopes of the broken lines correspond to omni-directional energy spectra $\sim k^{-\frac{5}{3}}$ and k^{-3} , respectively. Fig. 2c is a plot of the "dissipation spectrum" $k^4 |\underline{v}(\underline{k})|^2$. The broken lines correspond to slopes for omni-directional energy spectra $\sim k^{-\frac{5}{3}}$ and k^{-3} , respectively.
- Fig. 3 Enstrophy $\Omega(t)$ and energy $\epsilon(t)$ vs. time for MHD run with mechanical forcing only ($g \equiv 0$). Note the qualitative similarity to the pure Navier-Stokes case shown in Fig. 1. $\nu = \mu = 10^{-3}$.

Fig. 4 Modal energies $|\underline{v}(k)|^2$ vs. k^2 for an MHD run with mechanical forcing only (same situation as Fig. 3). The lines have slopes corresponding to omni-directional energy spectra $\sim k^{-\frac{5}{3}}$ and k^{-3} , respectively. The modal energies have been averaged over 1200 time steps ending at $t = 75.0$. Note the similarities to the pure Navier-Stokes run (Fig. 2).

Fig. 5 Modal energies $|\underline{B}(k)|^2$ vs. k^2 for an MHD run with mechanical forcing only (same run as Figs. 3 and 4) and a small initial seed field $|\underline{B}(k)|^2 = 10^{-6}$, all k . The modal energies have been time averaged over 1200 time steps ending at $t = 75.0$. This evolution to a sharply rising magnetic spectrum is characteristic of all runs of this type.

Fig. 6(a) Modal energies $|\underline{B}(k)|^2$ vs. k^2 for a $\mu = \nu = 10^{-3}$ MHD run with magnetic forcing only ($f \equiv 0$). These values are instantaneous values at $t = 75.0$, and their comparison with the time averages in Fig. 6(b) shows a typical level of fluctuation-reduction achieved by the time averaging. The broken line drawn below the forcing band has a slope corresponding to an omni-directional magnetic energy spectrum $\sim k^{-\frac{1}{3}}$.

(b) Modal energies $|B(k)|^2$ vs. k^2 for the same MHD run shown in Fig. 6(a), but time averaged over 1200 time steps ending at $t = 75.0$. The broken line corresponds to an omni-directional magnetic energy spectrum $\sim k^{-3}$.

(c) Modal energies $|v(k)|^2$ vs. k^2 corresponding to Fig. 6(b), again time averaged over 1200 time steps ending at $t = 75.0$.

Fig. 7(a) Total energy $\epsilon(t)$ vs. time for the MHD run ($f = 0$) shown in Fig. 6. For an energy cascade, this number would saturate and become approximately constant in time.

(b) Cross-helicity P versus time.

(c) Mean square vector potential $A(t)$ vs. time corresponding to the MHD run ($f = 0$) shown in Fig. 6. A pure inverse vector potential cascade would appear as a monotonic increase of A with time.

Fig. 8 Total energy $\epsilon(t)$, cross helicity $P(t)$, and mean square vector potential $A(t)$ vs. time for an MHD run with uncorrelated mechanical and magnetic forcing ($f \neq 0, g \neq 0$) of approximately equal magnitude. For a dual-cascade situation in which no cross-helicity was being added, $\epsilon(t)$ would saturate, $P(t)$ would fluctuate close to zero, and $A(t)$ would increase linearly.

$$\mu = \nu = 10^{-3}.$$

Fig. 9 Total mechanical energy, magnetic energy and ratio R vs. time for the forced MHD run shown in Figs. 8 and 10.

Fig. 10(a) Modal energies $|\tilde{B}(k)|^2$ and (b) modal energies $|\tilde{v}(k)|^2$ for the forced ($f \neq 0$, $g \neq 0$) MHD run shown in Figs. 8 and 9. Substantial back-transfer of magnetic excitation into the longest wavelength modes has occurred, but the fluctuations are larger than for either the $f = 0$ or $g = 0$ cases. Both (a) and (b) represent time averages over 720 time steps ending at $t = 71.0$.

Fig. 11(a) Modal energies $|\tilde{B}(k)|^2$ and (b) modal energies $|\tilde{v}(k)|^2$ vs. k^2 for a forced ($f \neq 0$, $g \neq 0$) MHD run which is the same as Figs. 8, 9, 10, except that the dissipation has been increased to $\mu = \nu = 10^{-2}$. Averages have been taken over 1200 time steps ending at $t = 62.25$. The forcing band modal energies are off scale, being $\gtrsim 1.4 \times 10^{-3}$. The most unusual feature of Fig. 11 is the separation of the mechanical and magnetic energies at the highest k values. (Neither ϵ nor A increases for this case after about $t = 3.0$.)

Fig. 12 Dissipation spectra, (a) $k^2 |\tilde{v}(k)|^2$ and (b) $k^2 |\tilde{B}(k)|^2$, for the MHD run ($f \neq 0$, $g \neq 0$) shown in Figs. 8, 9, 10. The pure double-cascade theory would predict two straight lines for $|\tilde{B}(k)|^2$ vs. k^2 , the one below the forcing band with a slope of $\frac{2}{3}$ and the one above the forcing wave number with a slope of either $1/6$ or $1/4$, on a $\ln k^2 |\tilde{B}(k)|^2$ vs.

$\ln k^2$ plot. Neither region is fit very well by these straight lines. If $k_{\max}/k_{\min}$ were much larger, this spectrum should eventually reach a maximum and decrease sharply with increasing k^2 .

Fig. 13(a) Magnetic modal energies, and (b) kinetic modal energies vs. k^2 for a forced MHD run ($f \neq 0$, $g \neq 0$) similar to Figs. 8, 9, 10, except that the magnitudes of f and g have been doubled. Both (a) and (b) are time averages over 1200 time steps ending at $t = 75.0$. Fig. (13a) shows the greatest back-transfer of magnetic energy to the $k^2 = 1$ modes of any of the runs.

Fig. 14(a) Modal energies, time averaged over 1200 time steps ending at $t = 37.5$ for pure Navier-Stokes initial-value problem with zero viscosity. The theoretical prediction is the solid line. The fractional variation over the duration of the run was 0.4% for the energy and 5.4% for the enstrophy.

(b) Modal energies for the velocity field for the same situation as Fig. (14a), but with an approximately 3% addition of magnetic excitation ($\mu = \nu = 0$). The run is time averaged over 1024 time steps ending at $t = 40.0$. The theoretical curve is the solid line (flat, in this case).

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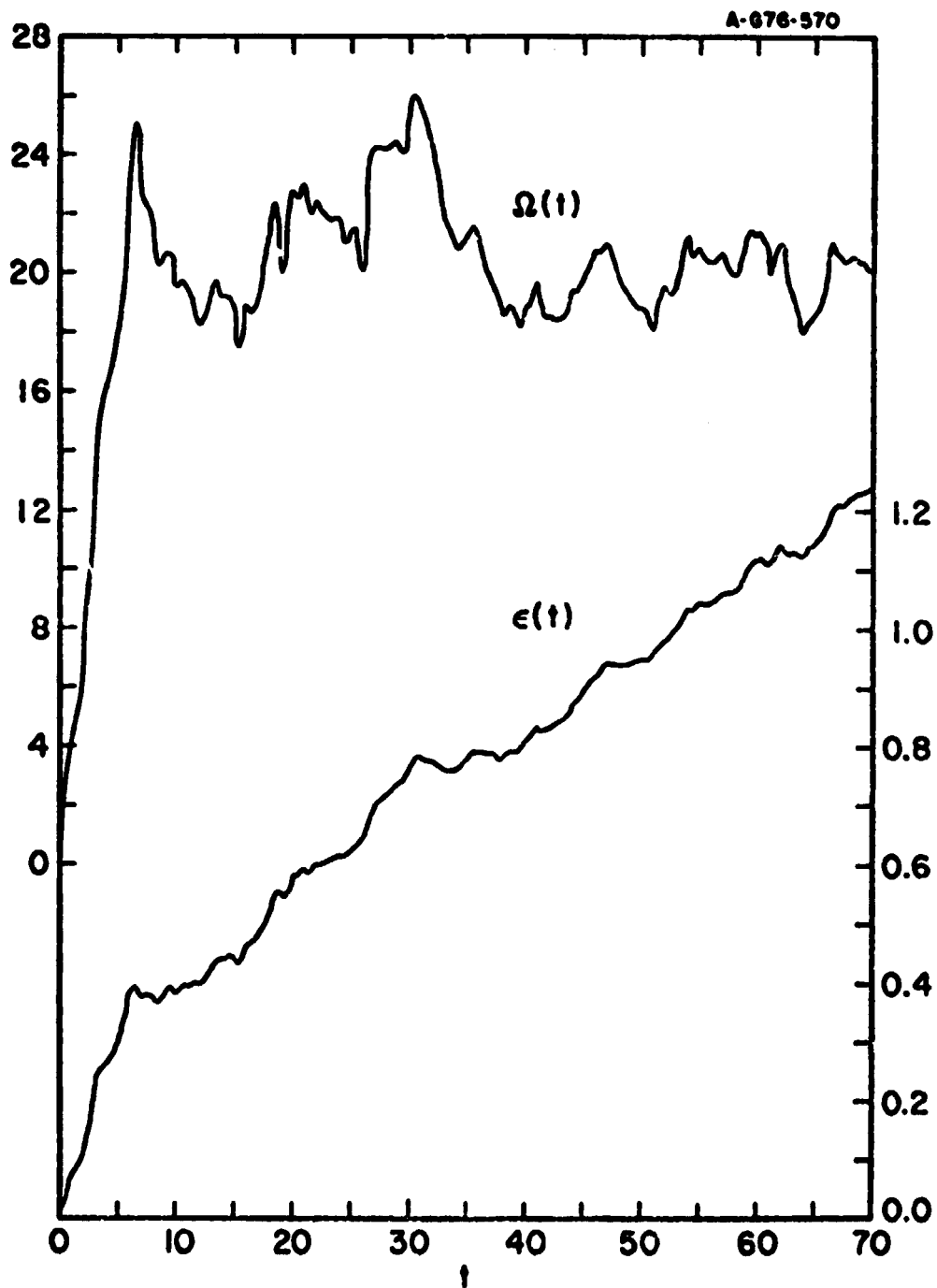


Figure 1

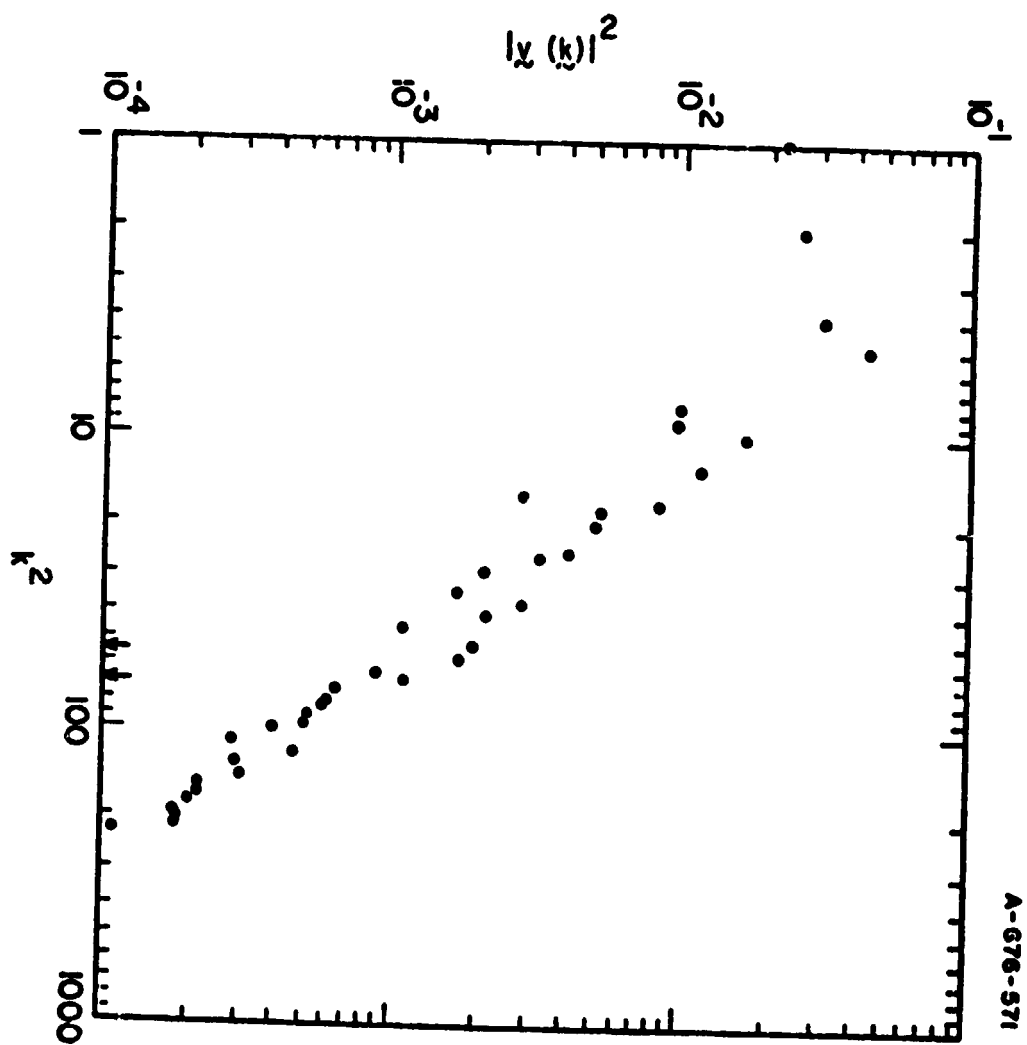


Figure 2a

A-676-572

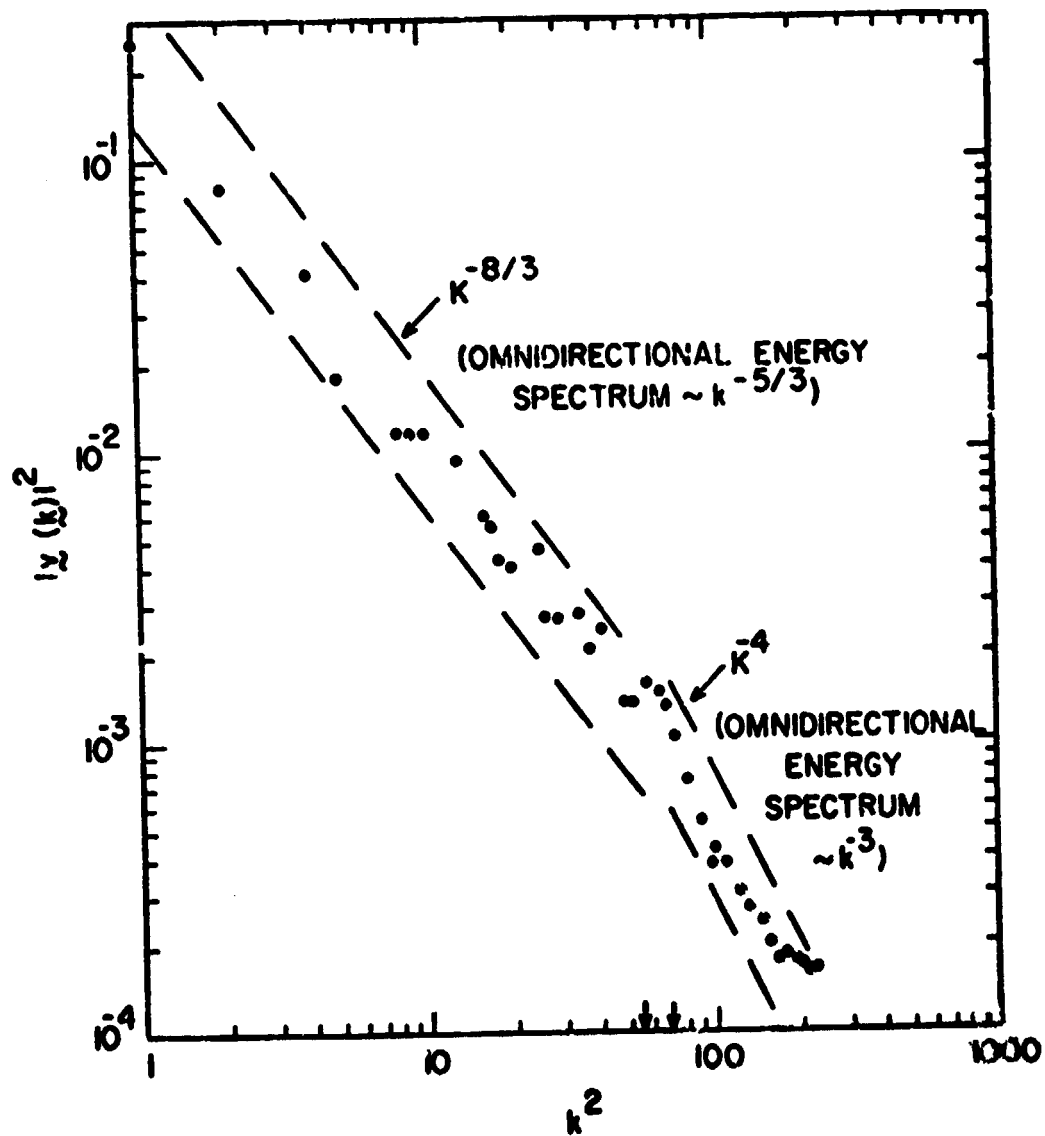


Figure 2b

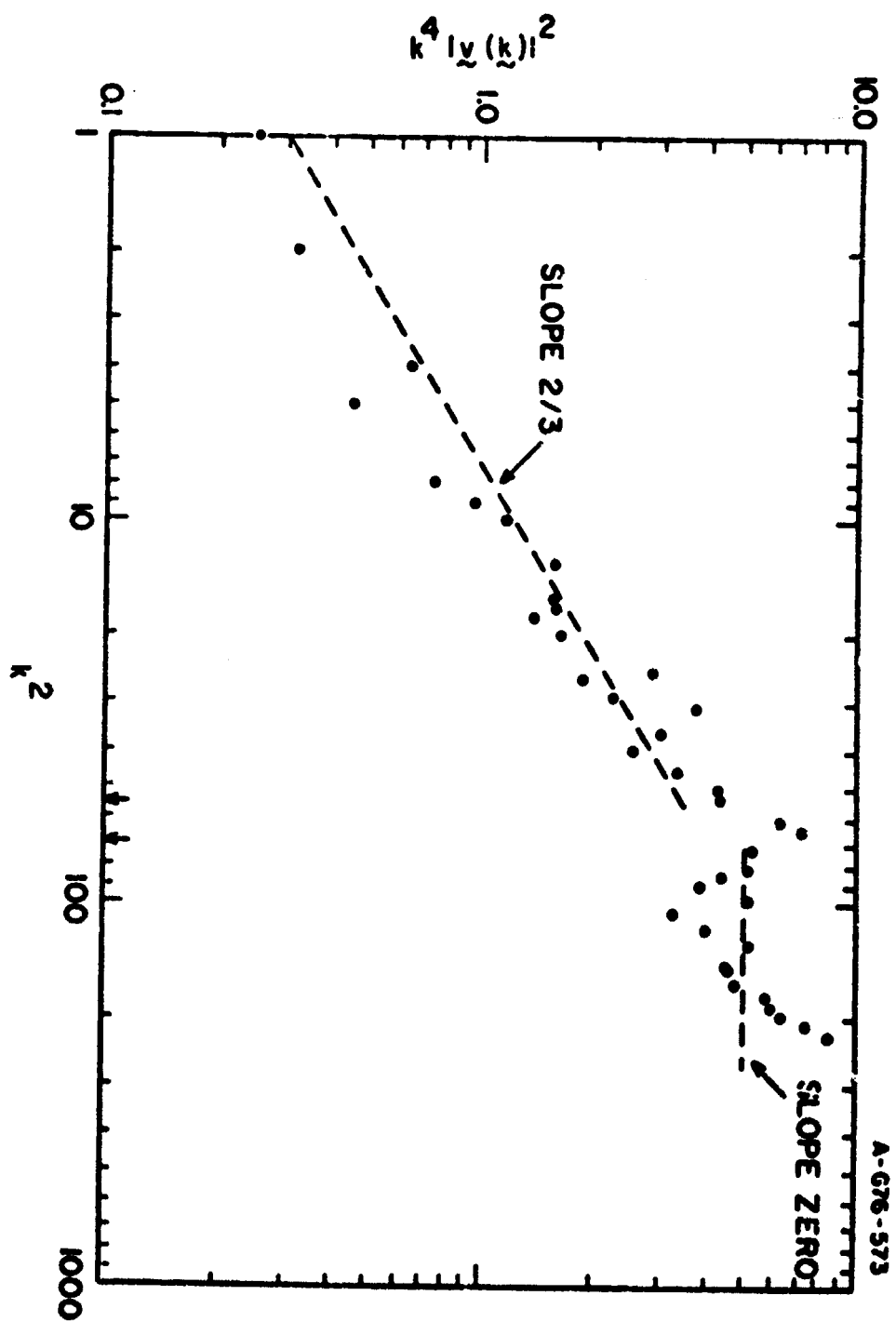


Figure 2c

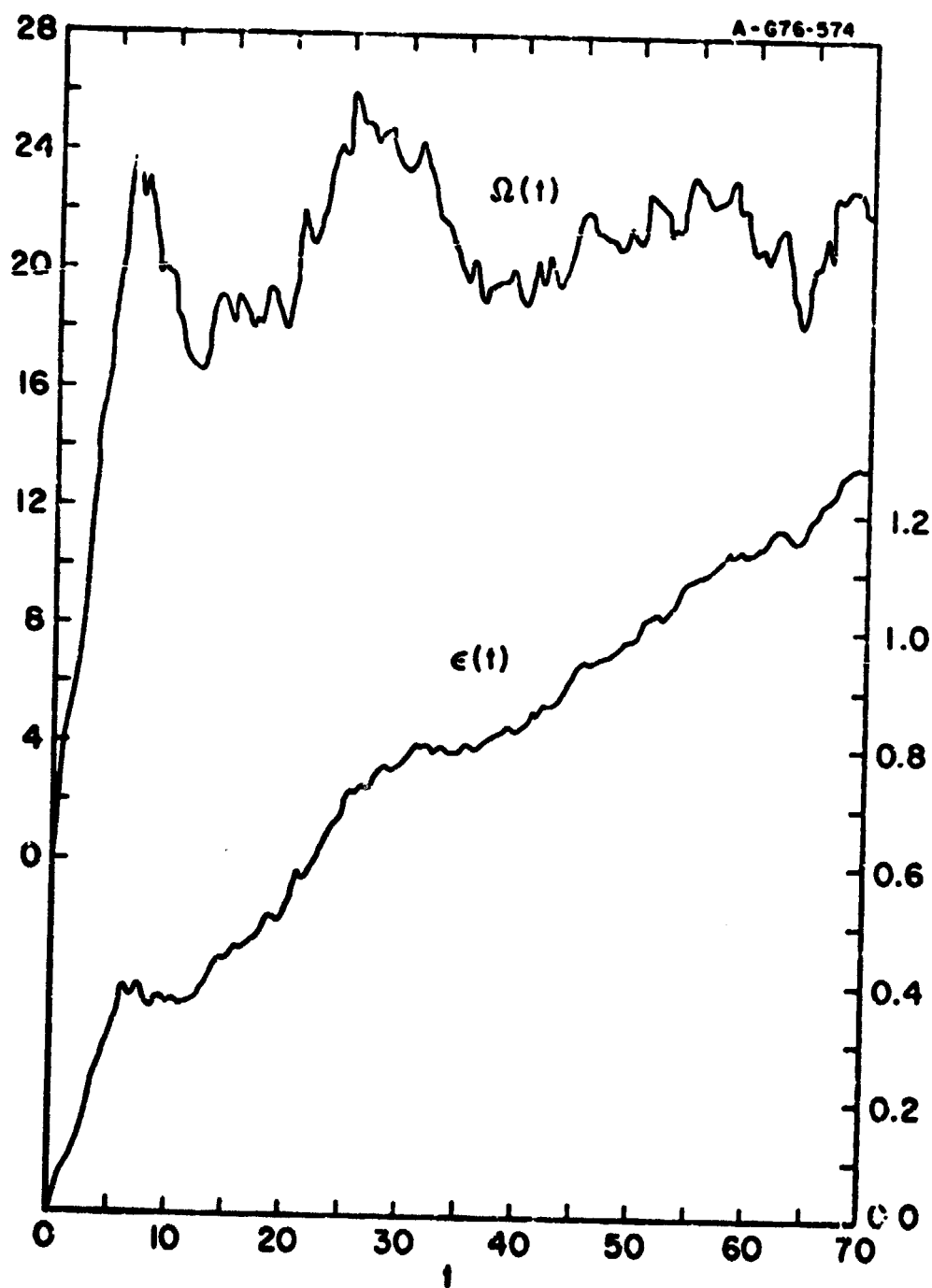


Figure 3

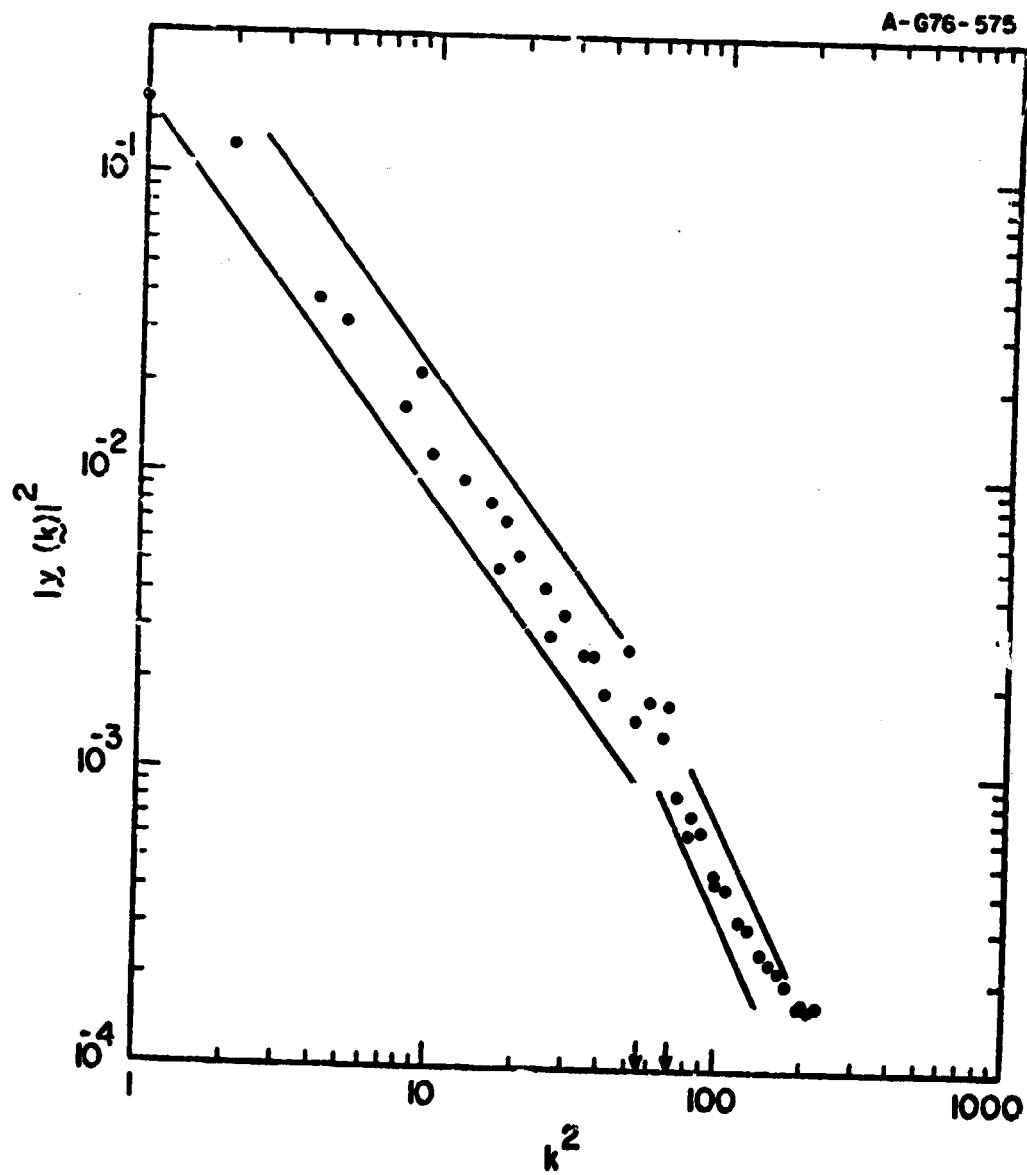


Figure 4

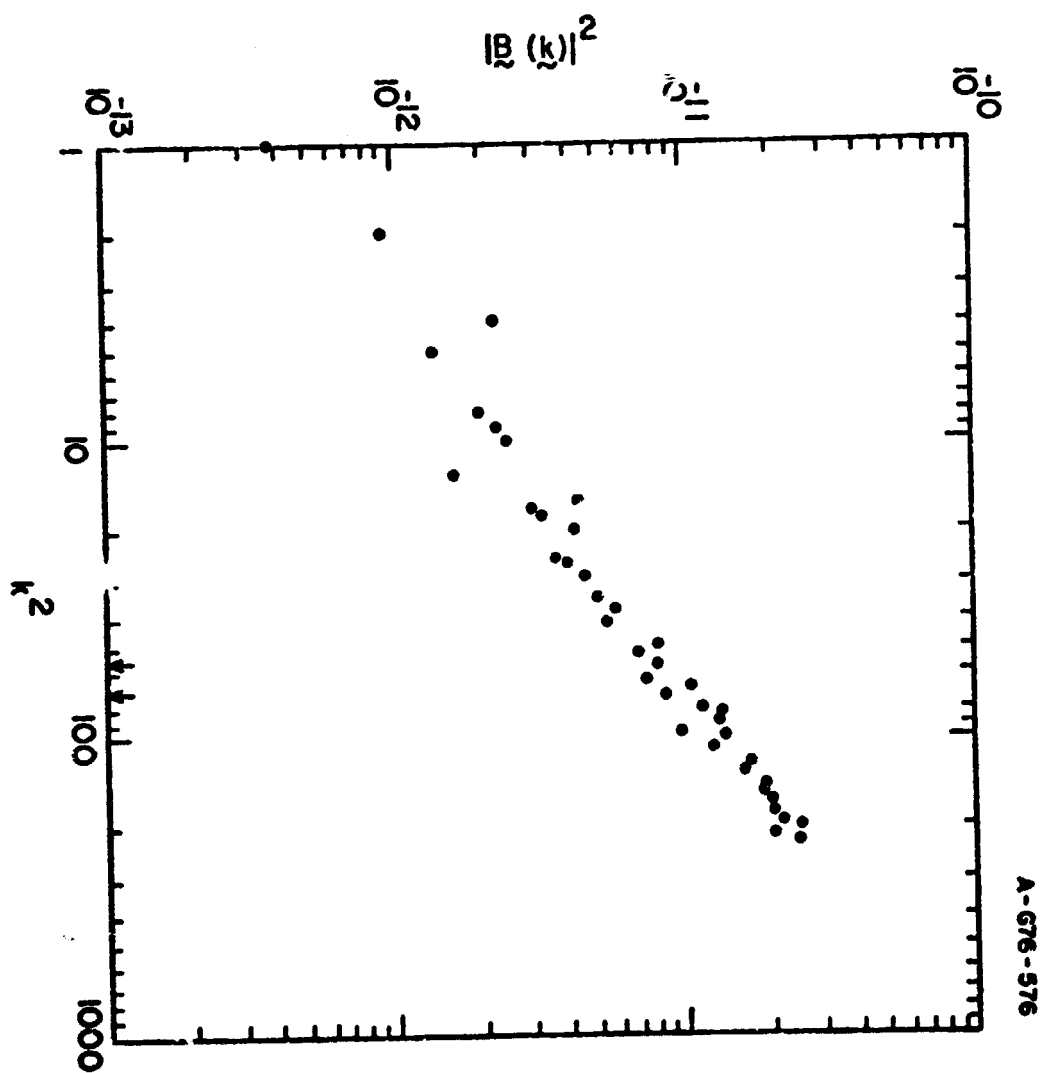


Figure 5

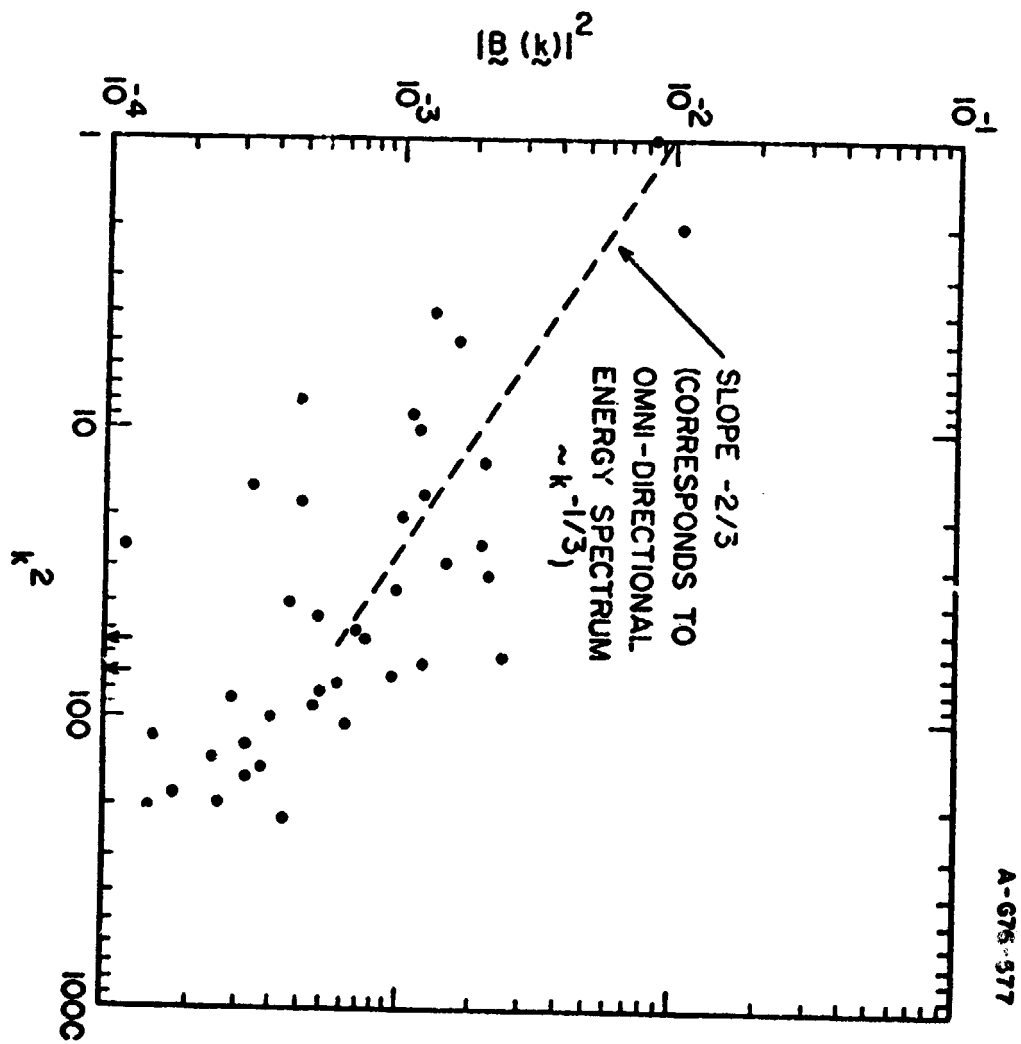


Figure 6a

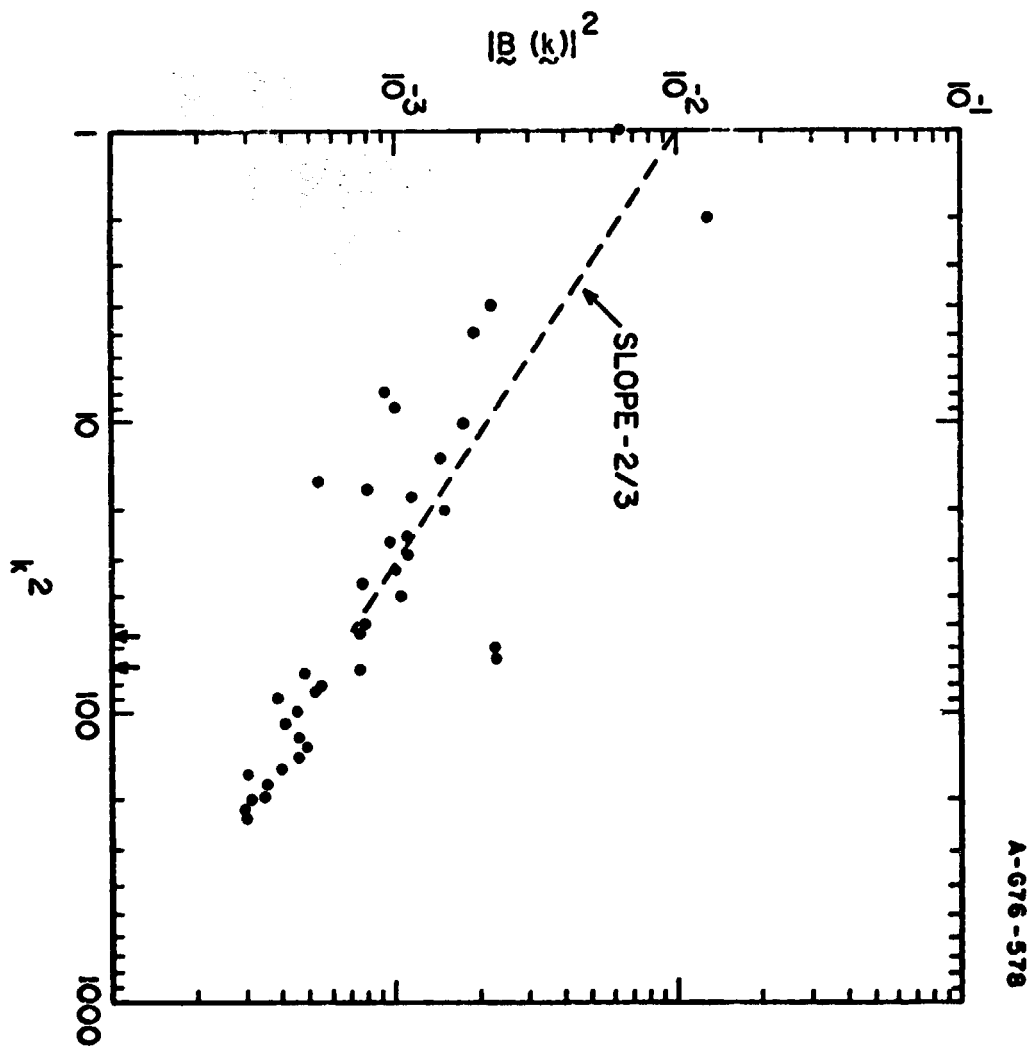


Figure 6b

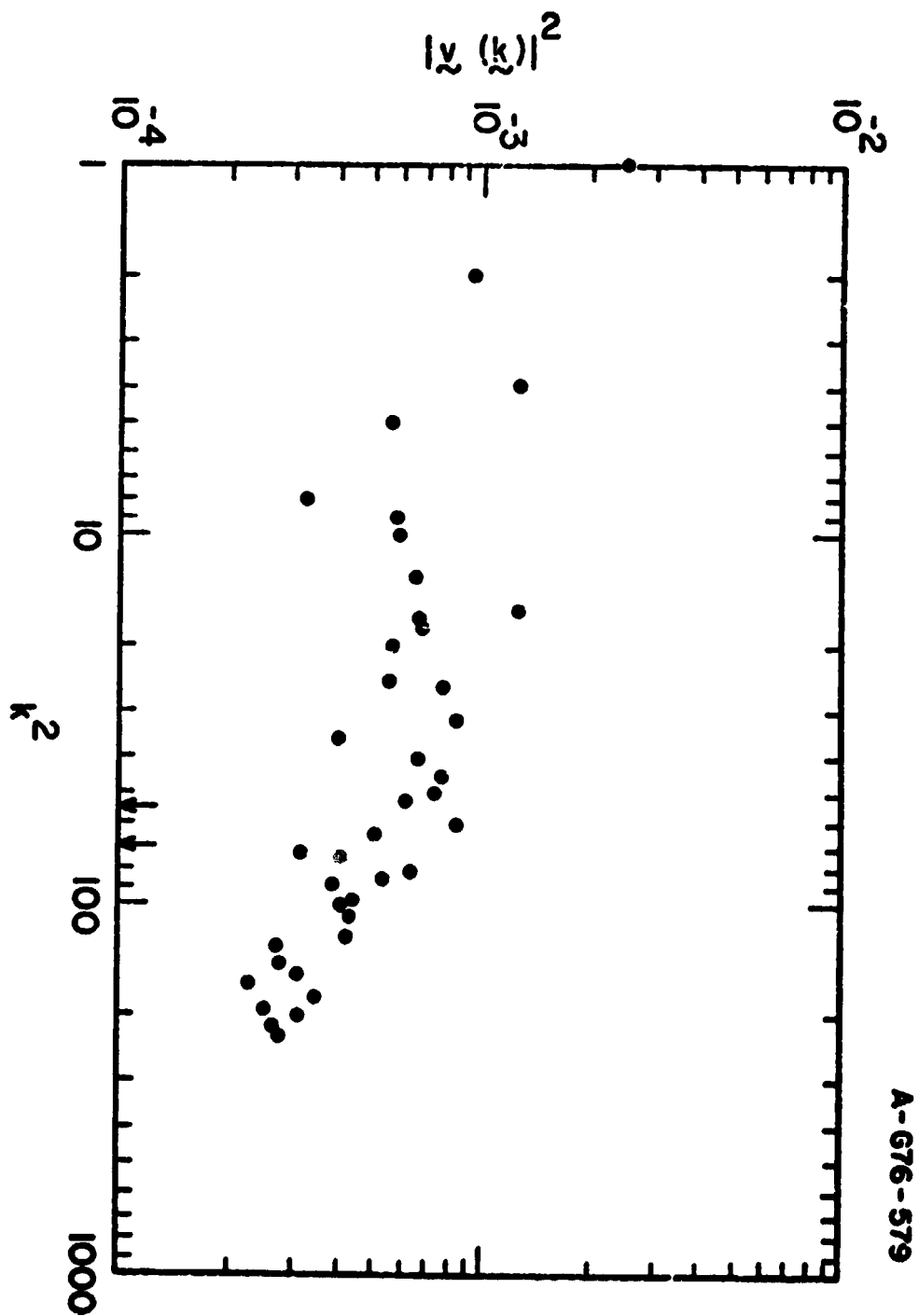


Figure 6c

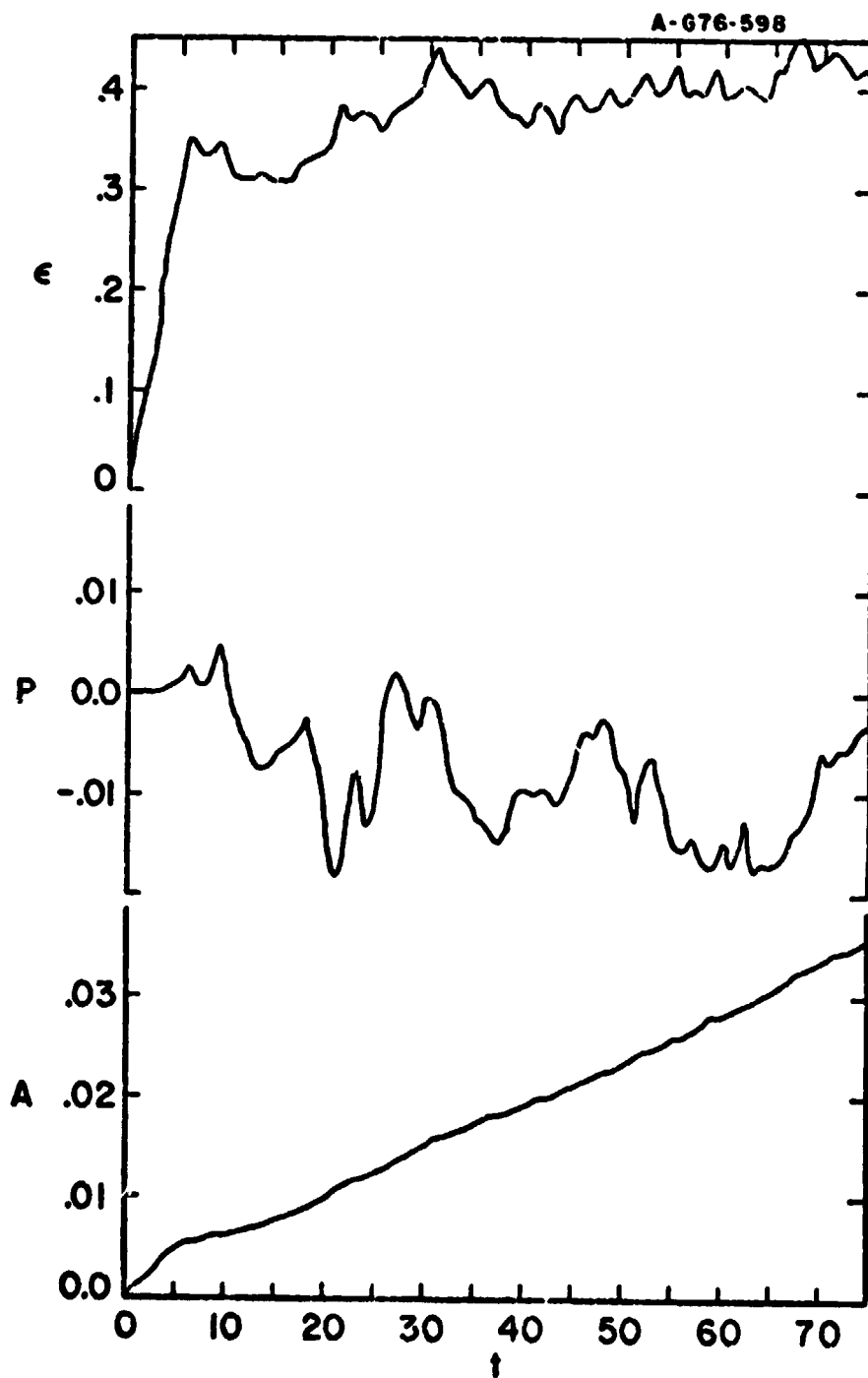


Figure 7a, b, c

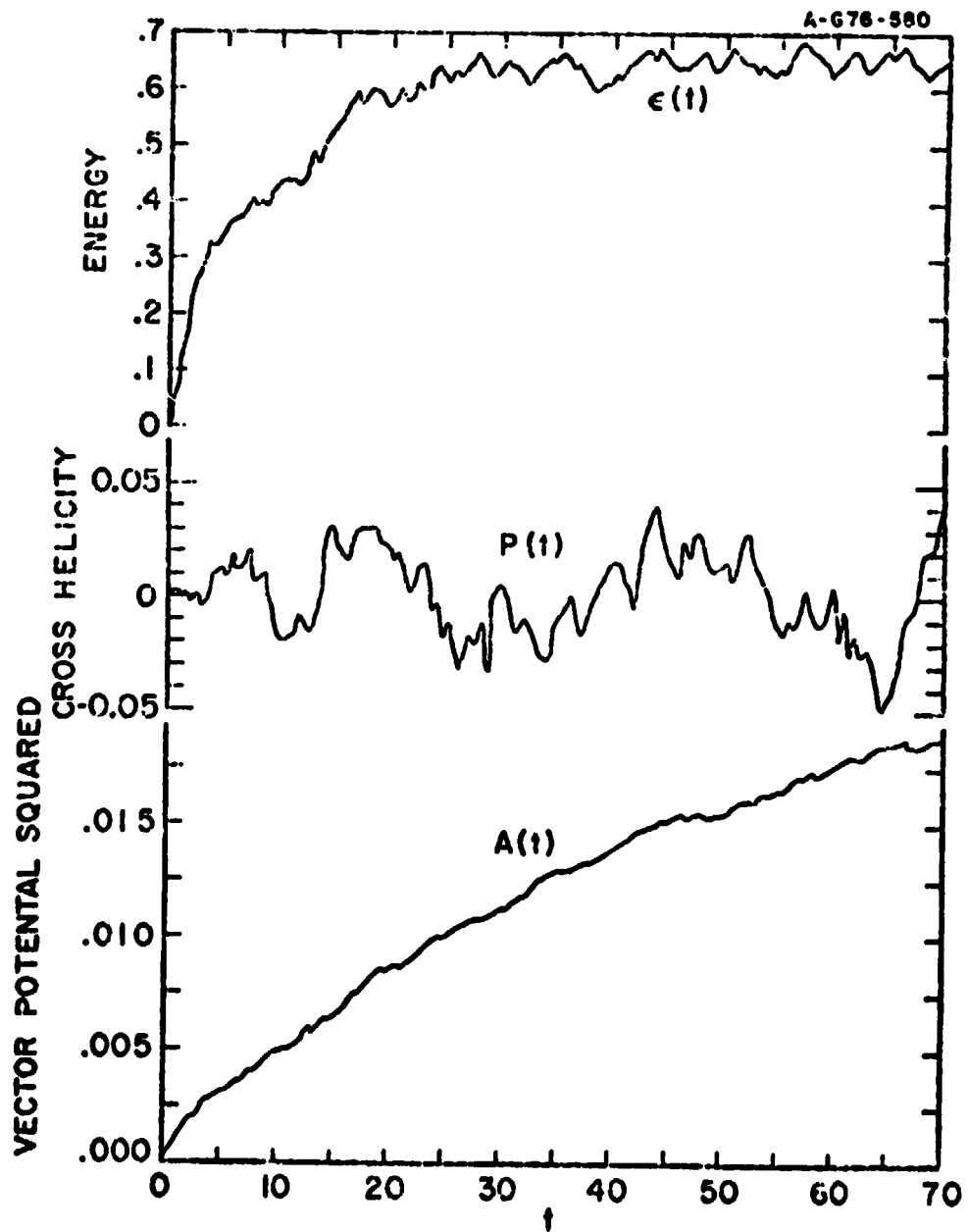


Figure 8

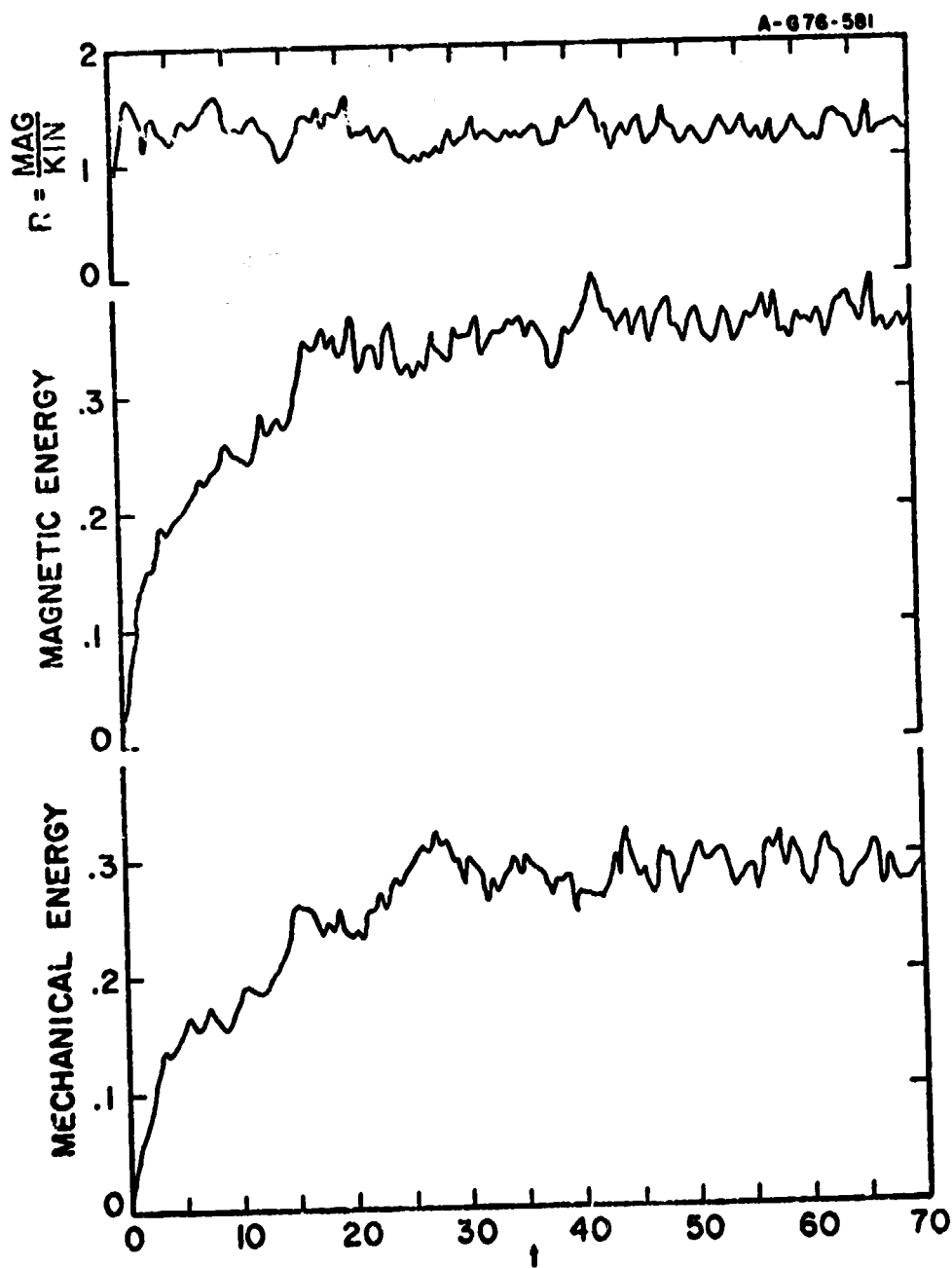
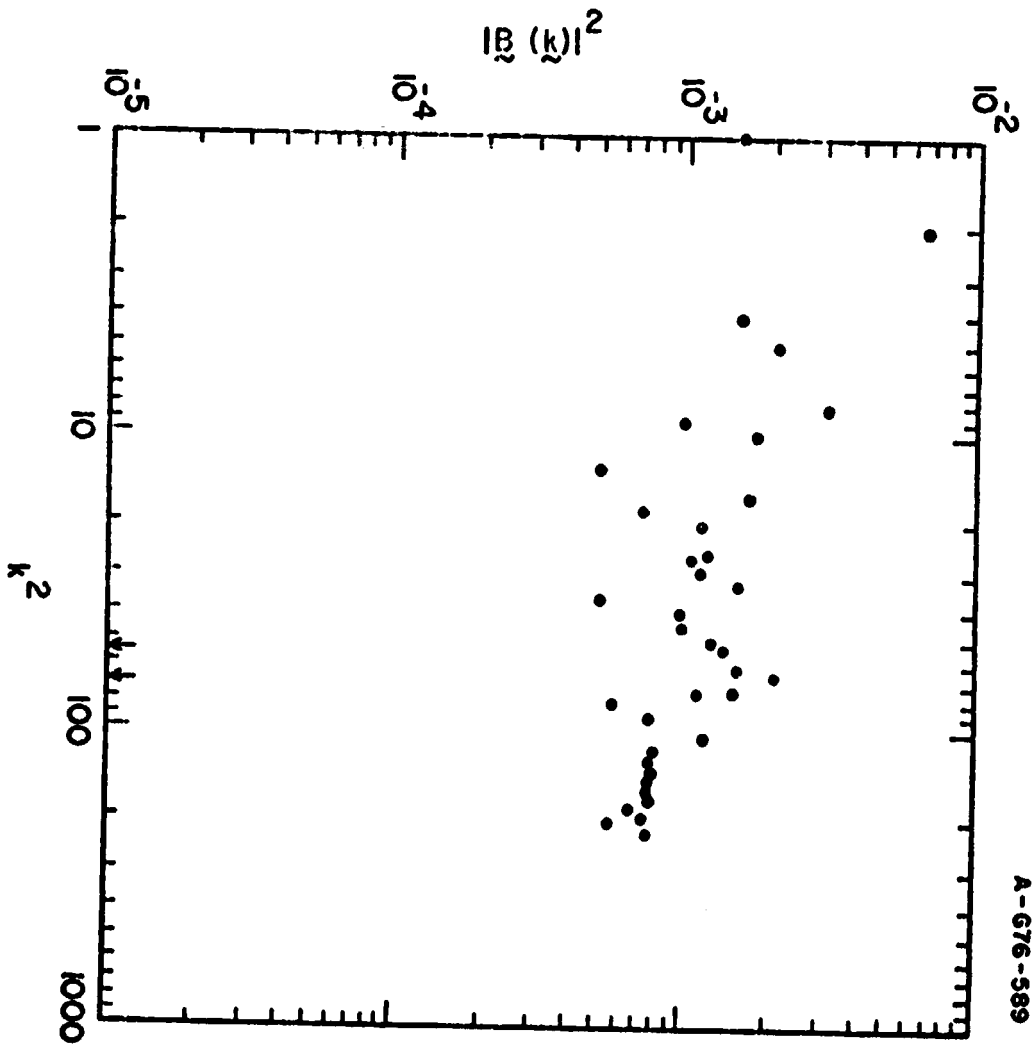


Figure 9



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Figure 10a

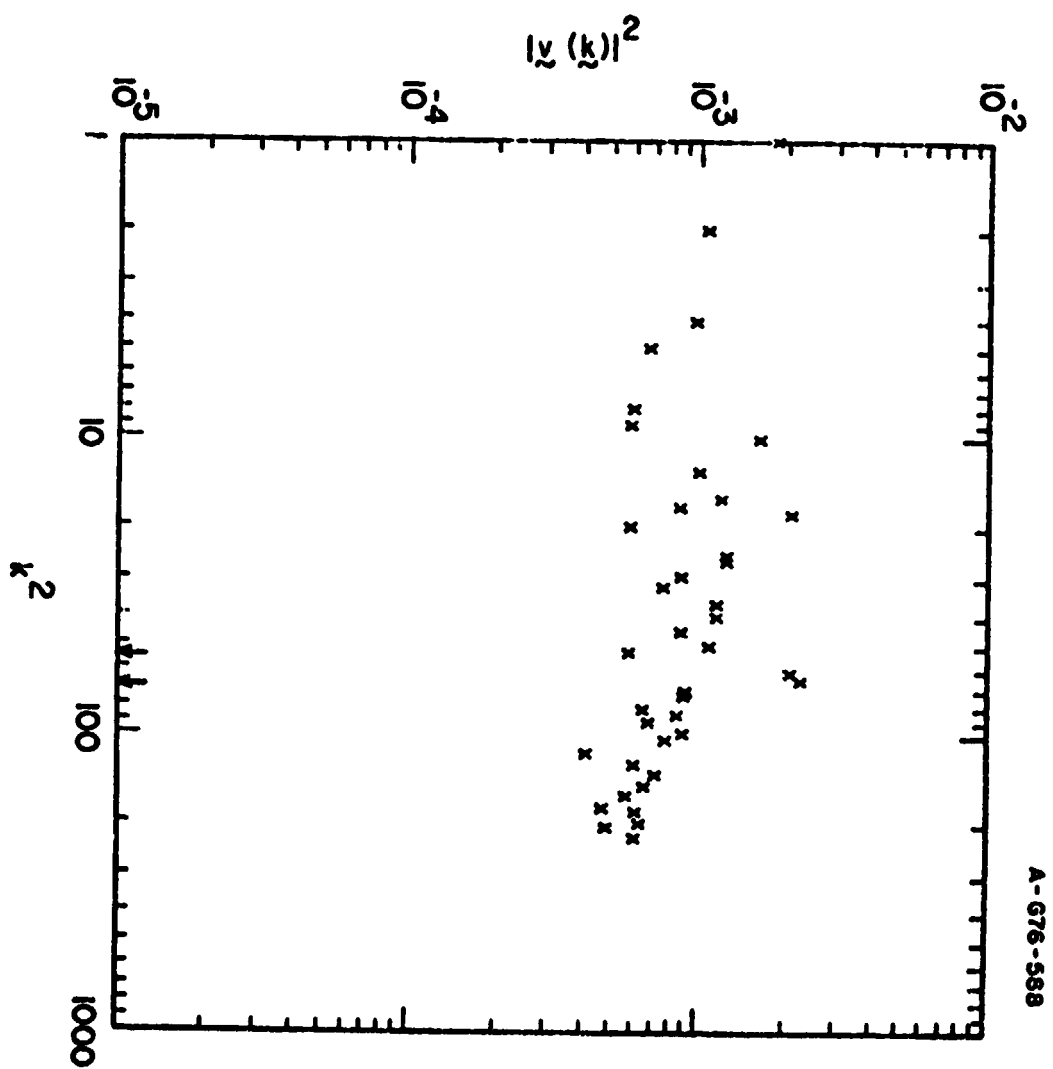


Figure 10b

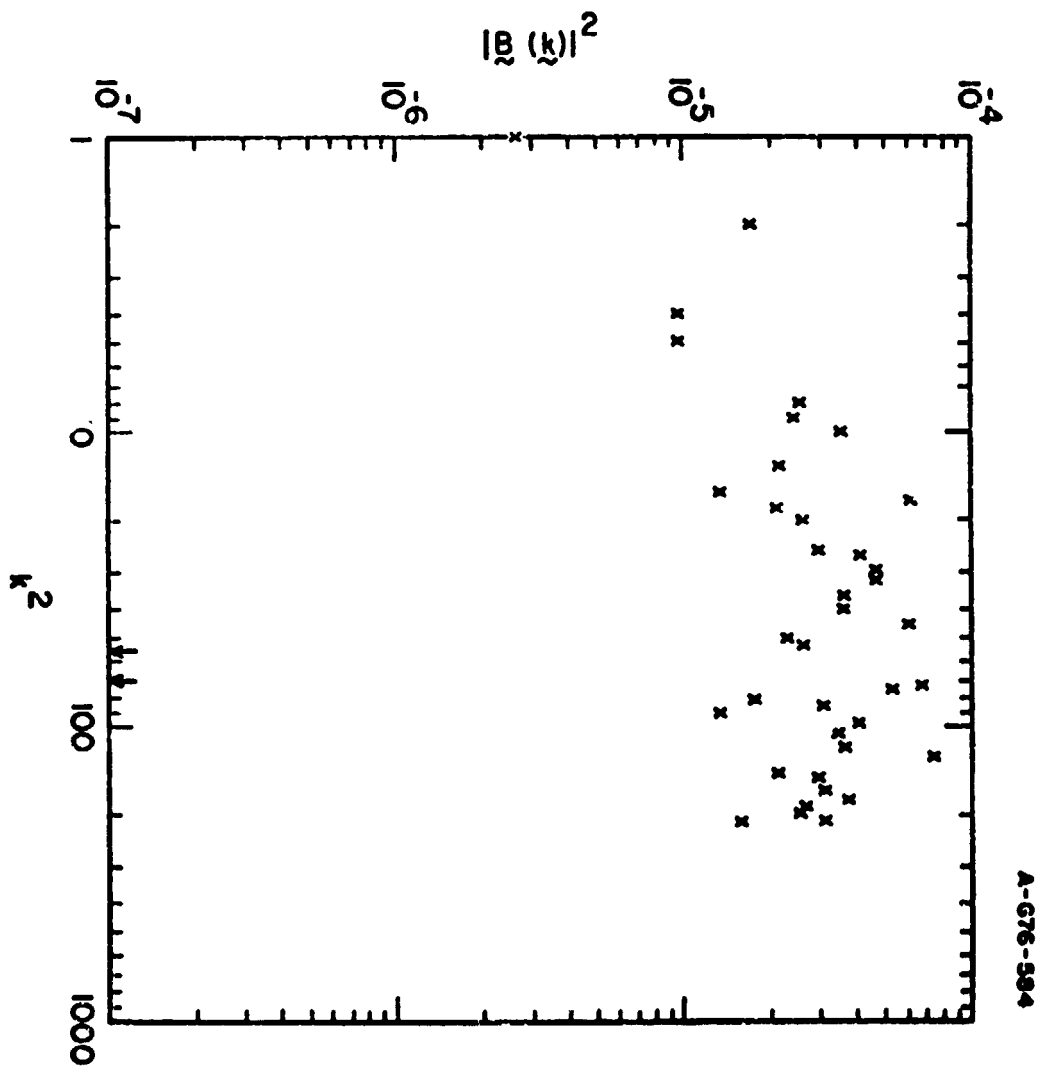
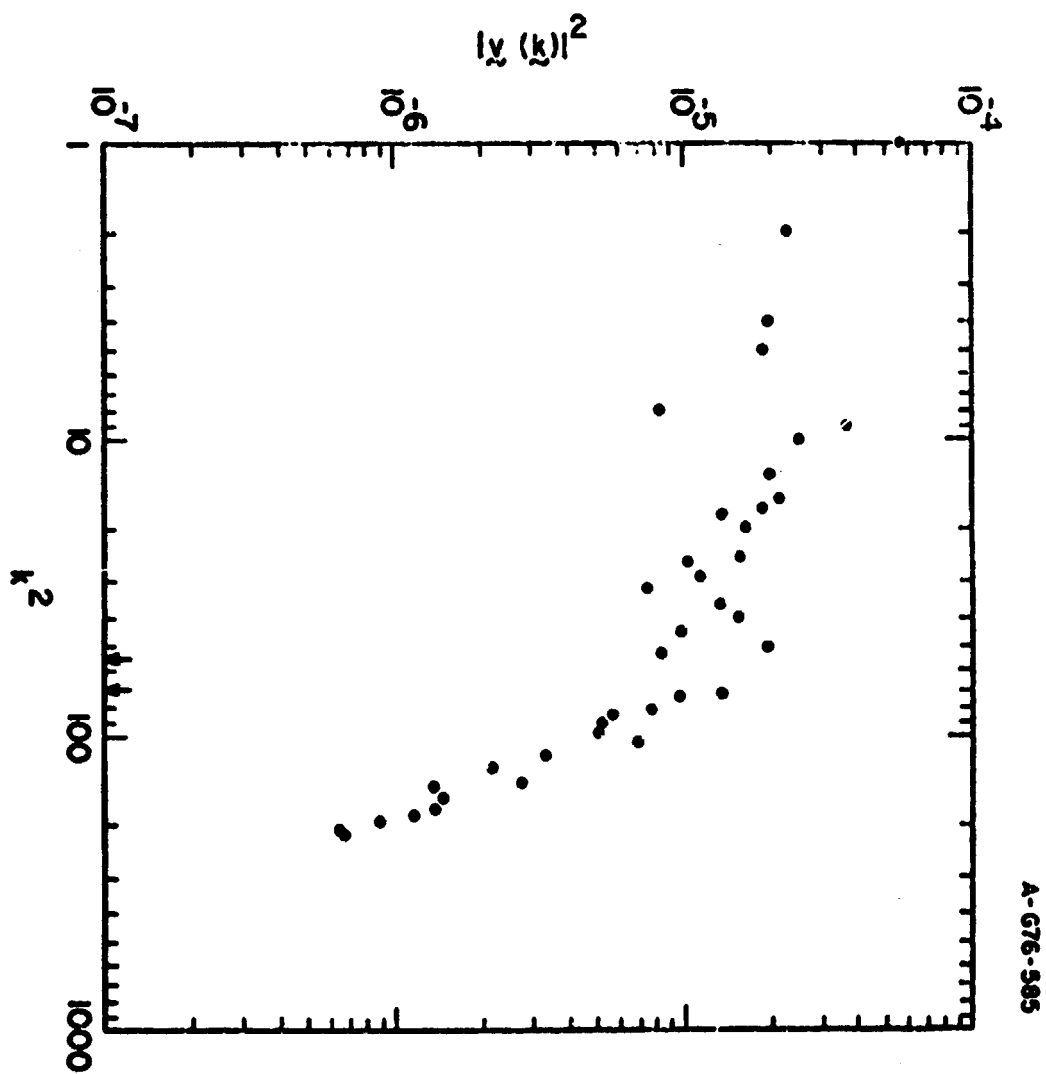


Figure 11a



A-676-585

Figure 11b

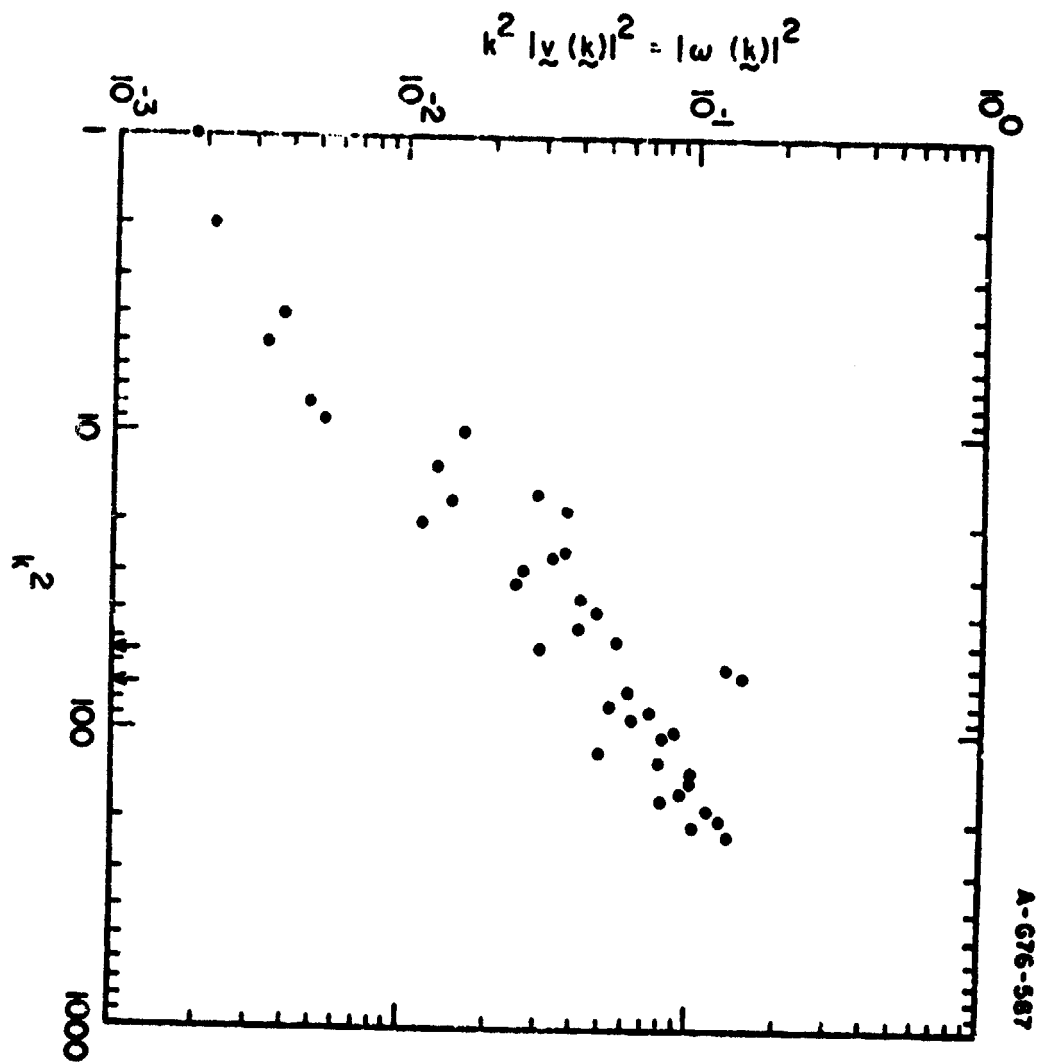


Figure 12a

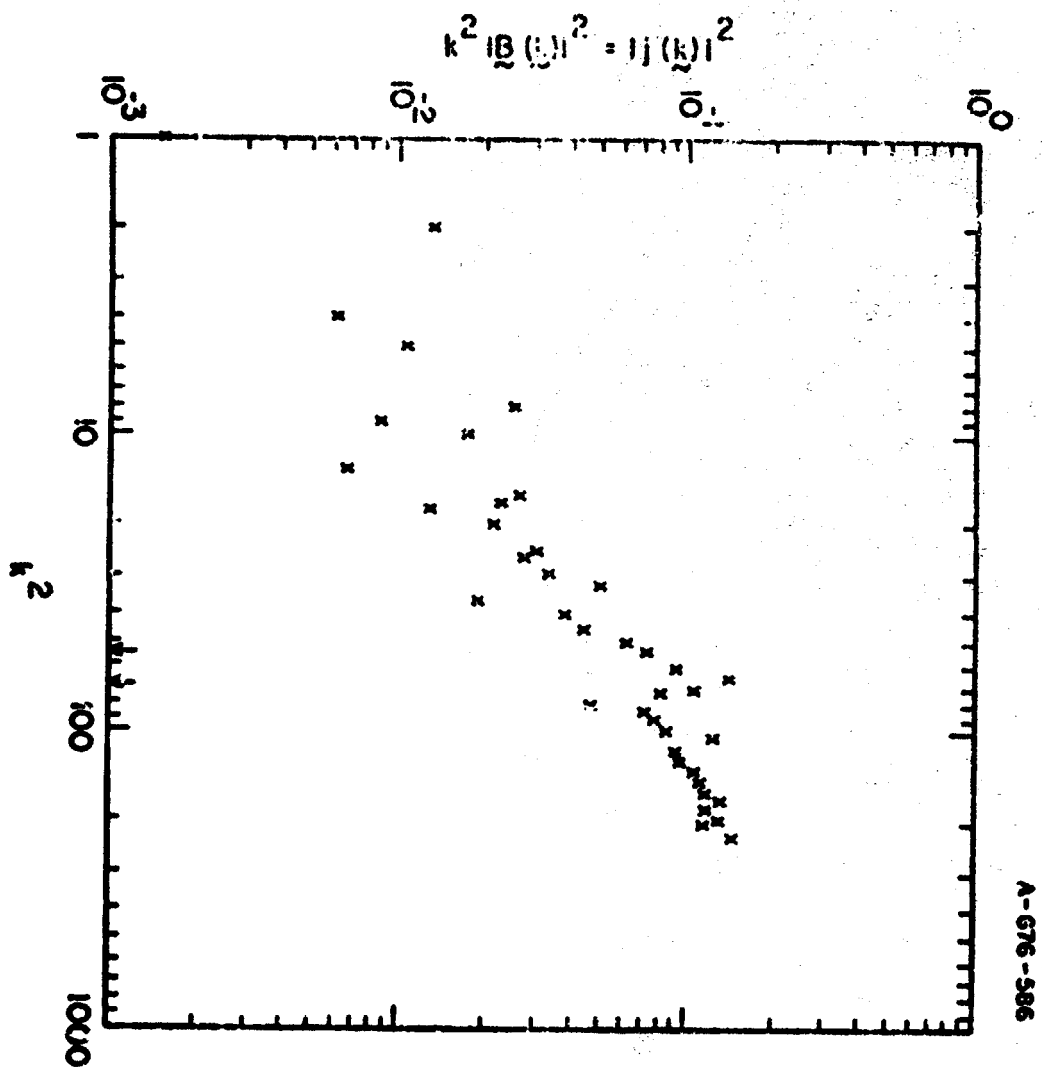


Figure 12b

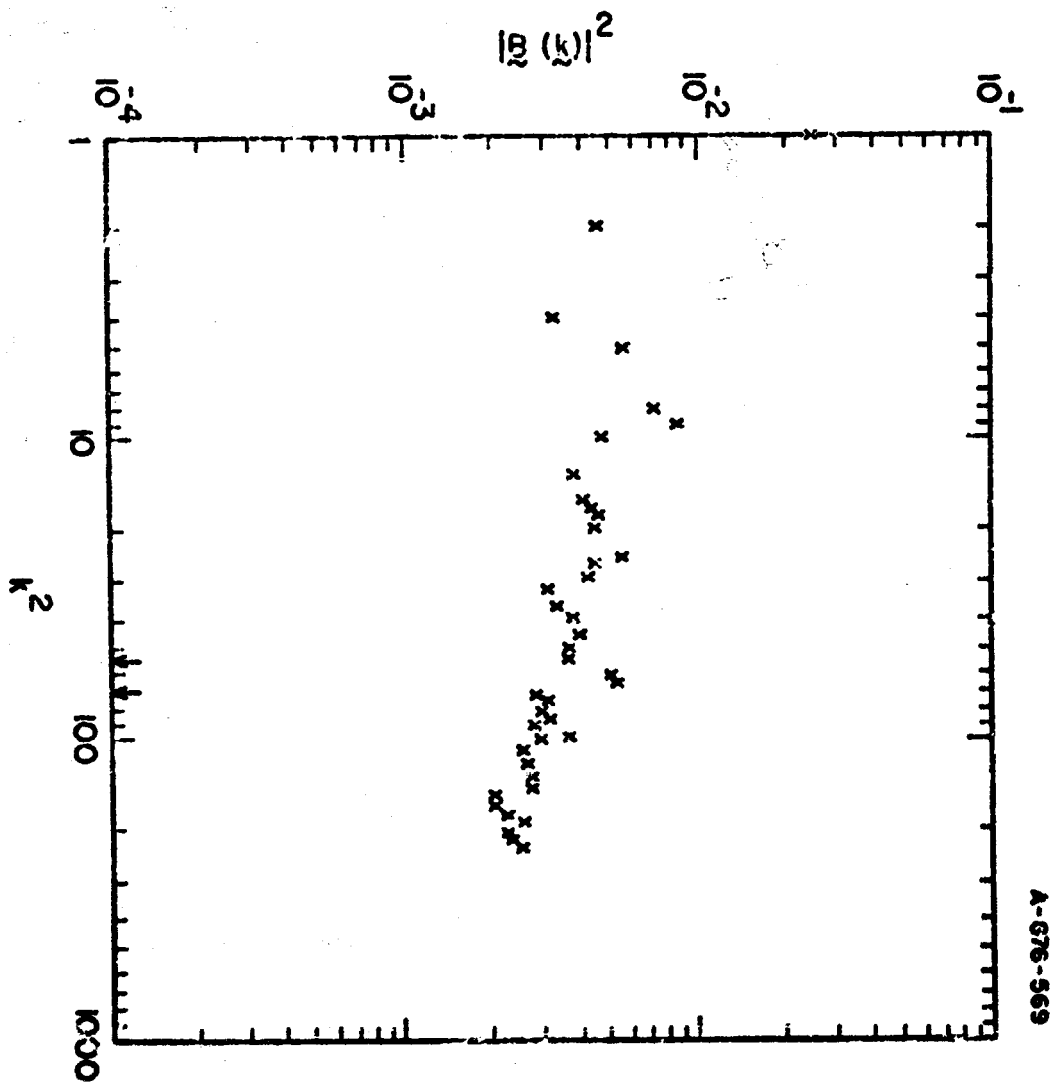


Figure 13a

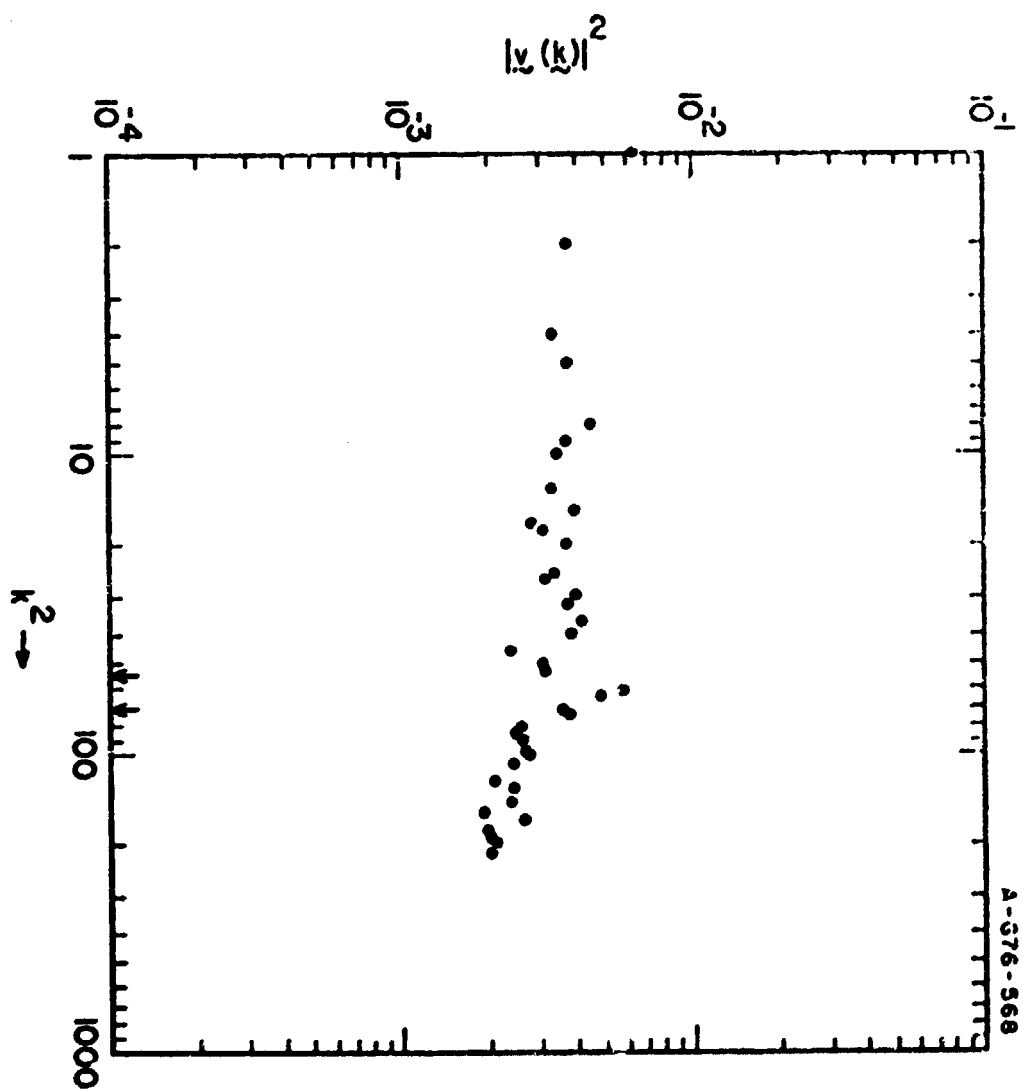


Figure 13b

A-676-612

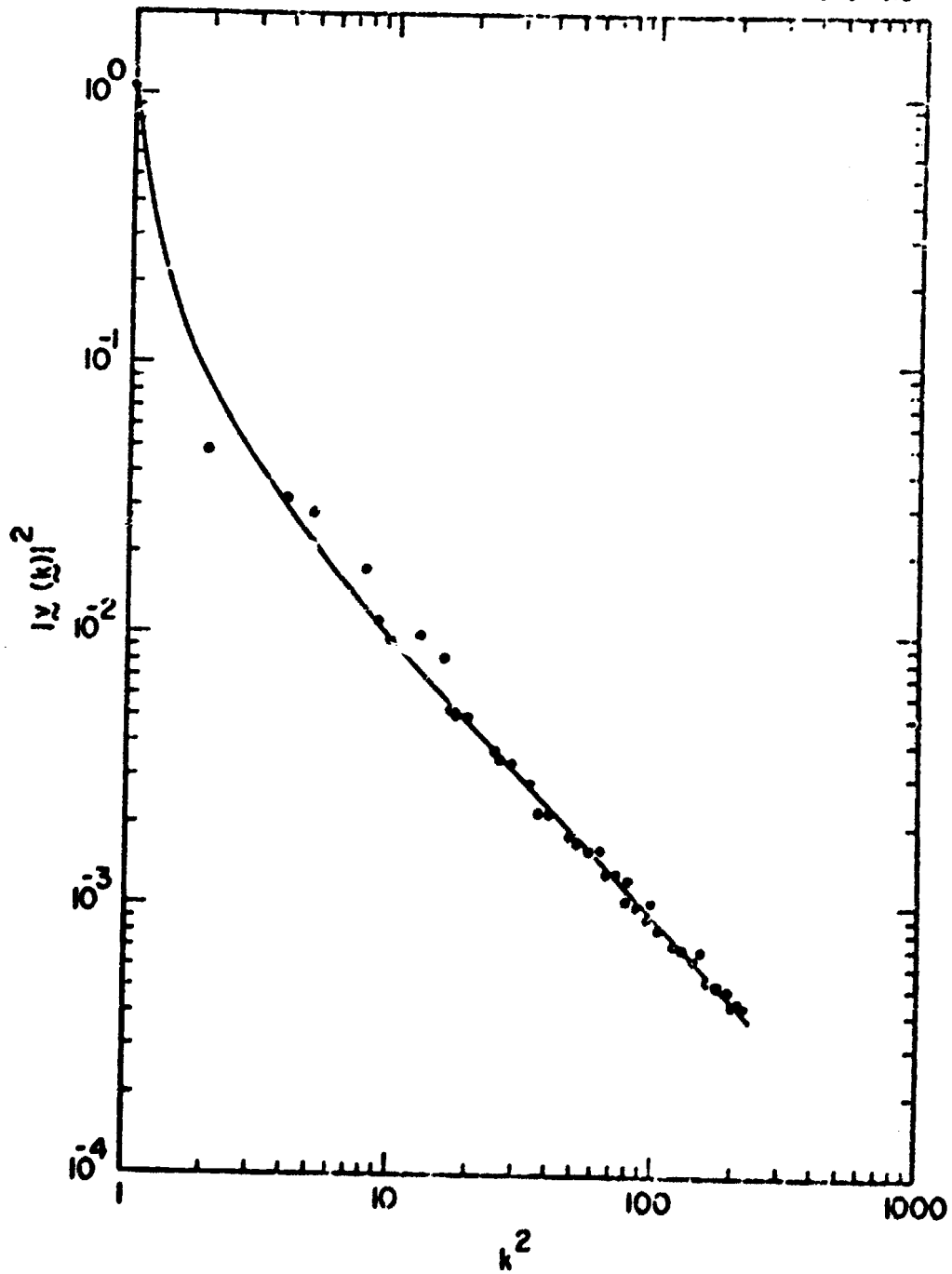


Figure 14a

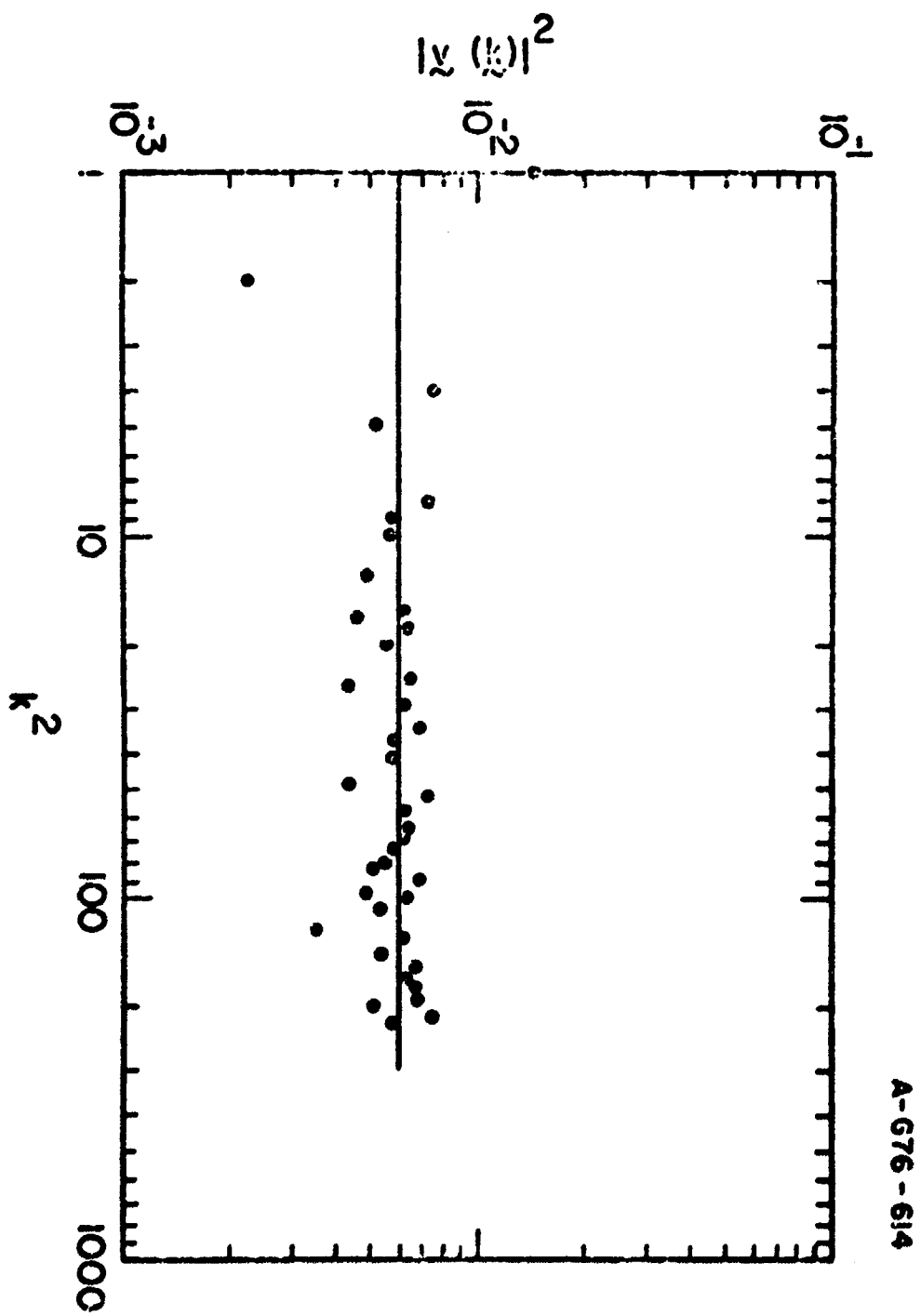


Figure 14b